SIAM J. Financial Math.
Vol. 5, pp. 137–152


# Optimal Order Scheduling for Deterministic Liquidity Patterns*

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**Abstract.** We consider a broker who has to place a large order which consumes a sizable part of average daily trading volume. The broker's aim is thus to minimize execution costs he incurs from the adverse impact of his trades on market prices. In contrast to the previous literature (see, e.g., Obizhaeva and Wang [A. Obizhaeva and J. Wang, *J. Financial Markets*, 16 (2013), pp. 1–32] and Predoiu, Shaikhet, and Shreve [*SIAM J. Financial Math.*, 2 (2011), pp. 183–212]), we allow the liquidity parameters of market depth and resilience to vary deterministically over the course of the trading period. The resulting singular optimal control problem is shown to be tractable by methods from convex analysis, and, under minimal assumptions, we construct an explicit solution to the scheduling problem in terms of some concave envelope of the resilience-adjusted market depth.

**Key words.** order scheduling, liquidity, convexification, singular control, convex analysis, envelopes, optimal order execution

**AMS subject classifications.** 91B26, 91G80, 90C20, 90C25

**DOI.** 10.1137/120897511

## 1. Introduction.

It is well known that market liquidity exhibits deterministic intraday patterns; see, e.g., Chordia, Roll, and Subrahmanyam [3] or Kempf and Mayston [6] for some empirical investigations. The academic literature on optimal order scheduling, however, mostly considers time-invariant specifications of market depth and resilience; cf. Obizhaeva and Wang [7], Alfonsi, Fruth, and Schied [2], and Predoiu, Shaikhet, and Shreve [8]. How to account for time-varying specifications of these liquidity parameters when minimizing the execution costs of a trading schedule thus becomes an issue.

Using dynamic programming techniques and calculus of variations, this problem was addressed by Fruth, Schöneborn, and Urusov [5]. These authors show that under certain additional assumptions on these patterns there is a time-dependent level for the ratio of the number of orders still to be scheduled and the current market impact which signals when additional orders should be placed. Explicit solutions are provided for some special cases where the broker is continually issuing orders. The thesis [4] discusses conditions under which the order signal structure persists in case of stochastically varying liquidity parameters. Acevedo and Alfonsi [1] use backward induction arguments in discrete time and then pass to continuous time to compute optimal policies for nonlinear specifications of market impacts which are scaled by a time-dependent factor satisfying some strong regularity conditions. In their approach order schedules are allowed in principle to sell and buy along the way, regardless

*Received by the editors November 2, 2012; accepted for publication (in revised form) October 21, 2013; published electronically February 25, 2014. Support of Einstein Foundation Berlin through grant "Game options and markets with frictions" and of DFG research center MATHEON is gratefully acknowledged.

http://www.siam.org/journals/sifin/5/89751.html

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of the sign of the desired terminal position, and they proceed to identify conditions (deemed to ensure absence of market manipulation strategies) under which optimal schedules will not do so. Optimal schedules are then obtained only under a strong assumption linking resilience and market depth to each other along with their time derivatives.

In contrast to these approaches, we focus from the outset on pure buying or selling schedules and show how to reduce our optimization problem to a convex one. Hence, we do not have to impose conditions ensuring that orders are scheduled in certain ways at certain times. Instead, optimal order sizes and times are derived endogenously from the structure of market depth and resilience alone. This is made possible by the use of convex analytic first-order characterizations of optimality which we show are intimately related to the construction of generalized concave envelopes of a resilience-adjusted form of market depth. Under minimal assumptions, this allows us to characterize when optimal schedules exist and, if so, to construct them explicitly in terms of these envelopes. We illustrate our findings by recovering the analytic solution of Obizhaeva and Wang [7], and we show how optimal schedules depend on fluctuations in market depth and the level of resilience. It turns out that with time-varying market depth optimal order schedules do not have to consist of big initial and terminal trades with infinitesimal ones in between as typically found in the previous literature. We also find that lower resilience will let optimal schedules focus more on (local) maxima of market depth to the extent that with no resilience optimal schedules trade only when market depth is at its global maximum.

**2. Set-up.** We consider a broker who has to place an order of a total number of $x > 0$ shares of some stock. The broker knows that, due to limited liquidity of the stock, these orders will be executed at a mark-up over some reference stock price. This mark-up will depend on the broker's past and present trades. For our specification of the mark-up we adopt the model proposed by Obizhaeva and Wang [7]; see also Alfonsi, Fruth, and Schied [2] and Predoiu, Shaikhet, and Shreve [8] for further motivation of this approach. In contrast to these papers, but in line with Fruth, Schöneborn, and Urusov [5] and Acevedo and Alfonsi [1], we will allow for the market's liquidity characteristics of depth and resilience to be changing over time according to a deterministic pattern.

Specifically, given the broker's cumulative purchases $X = (X_t)_{t\geq 0}$, a right-continuous increasing process with $X_{0-} \triangleq 0$, the resulting mark-up evolves according to the dynamics

$$\eta^X_{0-} \triangleq \eta_0 \geq 0, \quad d\eta^X_t = \frac{dX_t}{\delta_t} - r_t \eta^X_t \, dt, \tag{2.1}$$

where $\delta_t$ describes the market's depth at time $t \geq 0$ and where $r_t$ measures its current resilience. Thus, in our model market impact is taken to be a linear function of order size, the slope at any one time being determined by the market depth. Moreover, market impact decays over time at the rate specified by the market's resilience.

Clearly, (2.1) has the right-continuous solution

$$\eta^X_t \triangleq \left( \eta_0 + \int_{[0,t]} \frac{\rho_s}{\delta_s} \, dX_s \right) \Big/ \rho_t \text{ with } \rho_t \triangleq \exp\left( \int_0^t r_s \, ds \right), \quad t \geq 0, \tag{2.2}$$

under the following assumption.

Assumption 2.1. *The resilience pattern is given by a strictly positive and locally Lebesgue-integrable function* $r : [0, \infty) \to (0, \infty)$.

In what follows we shall furthermore require the following assumption.

Assumption 2.2. *The pattern of market depth* $\delta : [0, \infty) \to [0, \infty)$ *is nonnegative, not identically zero, bounded, and upper-semicontinuous with* $\limsup_{t\uparrow\infty} \delta_t/\rho_t = 0$.

The broker's aim is to minimize the cumulative mark-up costs:

$$\text{(2.3)} \qquad \text{Minimize } C(X) \triangleq \int_{[0,\infty)} \left(\eta^X_{t-} + \frac{\Delta_t X}{2\delta_t}\right) dX_t \text{ subject to } X \in \mathscr{X},$$

where $\Delta_t X \triangleq X_{t+} - X_{t-}$ and

$$\mathscr{X} \triangleq \{(X_t)_{t\geq 0} \text{ right-continuous, increasing } : X_{0-} = 0, X_\infty = x, C(X) < \infty\}$$

with the notation $X_\infty \triangleq \lim_{t\uparrow\infty} X_t$.

*Remark* 2.3. (i) Note that the $\frac{\Delta_t X}{2\delta_t}$-term in (2.3) accounts for the costs a noninfinitesimal order will incur due to its own mark-up effect; cf., e.g., Alfonsi, Fruth, and Schied [2] or Predoiu, Shaikhet, and Shreve [8], who in addition show how costs functionals as in (2.3) emerge with stochastic reference prices evolving as martingales when the broker is risk-neutral. Note also that, since we let $X_{0-} \triangleq 0$, a value of $X_0 > 0$ corresponds to an initial jump of size $\Delta_0 X = X_0$ in the order schedule.

(ii) To impose liquidation over a finite time horizon $T \geq 0$, one merely has to let the market depth $\delta_t = 0$ for $t > T$. Indeed, following the convention that $1/0 = \infty$ in the integration (2.2), $\eta^X$ and thus the costs $C(X)$ will then be infinite for any order schedule $X$ which increases after $T$.

(iii) Strict positivity of $r$ comes without loss of generality since if resilience $r = 0$ vanishes almost everywhere on an interval $[t_0, t_1]$, there is no need to trade it off against market depth there, and it is optimal to trade whatever amount is to be traded at the moment(s) when market depth $\delta$ attains its maximum over this period; cf. Proposition 4.1. Hence, $\delta$ could be assumed to take this maximum value at $t_0$, and the interval $(t_0, t_1]$ could then be removed from consideration.

(iv) The assumption of upper-semicontinuous market depth $\delta$ is necessary to rule out obvious counterexamples for existence of optimal schedules. For unbounded upper-semicontinuous $\delta$ one can easily show that $\inf_{\mathscr{X}} C = x\eta_0/\rho_\infty$, and so there is no optimal schedule. The lim sup condition is needed to rule out the optimality of deferring part of the order indefinitely.

(v) Including a discount factor with locally Lebesgue-integrable discount rate $\bar{r} = (\bar{r}_t) \geq 0$ in our mark-up costs is equivalent to considering $\widetilde{\delta}_t \triangleq \delta_t \exp(\int_0^t \bar{r}_s\, ds)$ and $\widetilde{r}_t \triangleq r_t + \bar{r}_t$, $t \geq 0$, instead of $\delta$ and $r$ above.

**3. Main result and sketch of its proof.** The main result of this paper is the solution to problem (2.3). It describes up to what mark-up level our broker should be willing to place orders at any point in time in order to minimize mark-up costs.

Theorem 3.1. *Suppose Assumptions 2.1 and 2.2 hold, let* $\lambda_t \triangleq \delta_t/\rho_t$ *and* $\widetilde{\lambda}_t \triangleq \sup_{u\geq t} \lambda_u$, *and define*

$$L^*_t = \inf_{u>t} \frac{\widetilde{\lambda}_u - \widetilde{\lambda}_t}{\widetilde{\lambda}_u/\rho_u - \widetilde{\lambda}_t/\rho_t}, \quad t \geq 0\,, \tag{3.1}$$

*where we follow the convention that* $0/0 \triangleq 0$.

*Then the optimal order schedule strategy is to place orders at any time* $t \geq 0$ *if and while the resulting mark-up is no larger than* $y^* L^*_t/\rho_t$, *i.e.,*

$$X^*_t = \lambda_0 (y^* L^*_0 - \eta_0)^+ + \int_{(0,t]} \lambda_s \, d \sup_{0\leq v\leq s} \left\{ (y^* L^*_v) \vee \eta_0 \right\}\,, \quad t \geq 0, \tag{3.2}$$

*provided the constant* $y^* > 0$ *in* (3.2) *can be chosen such that* $X^*_\infty = x$. *This is the case if and only if the right side of* (3.2) *with* $y^* \triangleq 1$ *remains bounded as* $t \uparrow \infty$. *If this is not the case, we have* $\inf_{X\in\mathscr{X}} C(X) = 0$, *and the problem does not have a solution.*

The following results outline the proof of this theorem and may be of independent interest. Our first auxiliary result provides a mathematically more convenient formulation of problem (2.3).

Proposition 3.2. *Suppose Assumptions 2.1 and 2.2 hold, let* $\lambda \triangleq \delta/\rho$ *and* $\kappa \triangleq \lambda/\rho = \delta/\rho^2$, *and define, for increasing and right-continuous* $Y = (Y_t)_{t\geq 0}$,

$$K(Y) \triangleq \frac{1}{2} \int_{[0,\infty)} \kappa_t \, d(Y^2_t)\,.$$

*Then*

$$Y_t = \eta_0 + \int_{[0,t]} \frac{dX_s}{\lambda_s},\ Y_{0-} \triangleq \eta_0, \text{ and } X_t = \int_{[0,t]} \lambda_s \, dY_s,\ X_{0-} \triangleq 0, \quad t \geq 0, \tag{3.3}$$

*define mappings from* $\mathscr{X}$ *to*

$$\mathscr{Y} \triangleq \left\{ (Y_t)_{t\geq 0} \text{ right-continuous, increasing } : Y_{0-} \triangleq \eta_0, \int_{[0,\infty)} \lambda_t \, dY_t = x, K(Y) < \infty \right\}$$

*and vice versa such that*

$$C(X) = K(Y)\,.$$

As a result, with these choices of $\kappa$ and $\lambda$, optimization problem (2.3) is equivalent to the following problem:

$$\text{Minimize } K(Y) \triangleq \frac{1}{2} \int_{[0,\infty)} \kappa_t \, d(Y^2_t) \text{ subject to } Y \in \mathscr{Y}. \tag{3.4}$$

Neither (2.3) nor (3.4) is convex in general.

Proposition 3.3. *For upper-semicontinuous* $\kappa$, *the functional* $K = K(Y)$ *of* (3.4) *is (strictly) convex for right-continuous increasing* $Y$ *with* $Y_{0-} = \eta_0$ *if and only if* $\kappa$ *is (strictly) positive and (strictly) decreasing.*

Convexity can always be arranged for, though, in the following sense.

Theorem 3.4. *Let $\lambda$, $\kappa$ be as in Proposition 3.2. Then optimization problem (3.4) has the same value as the convex optimization problem*

$$\text{(3.5)} \qquad \text{Minimize } \widetilde{K}(\widetilde{Y}) \triangleq \frac{1}{2}\int_{[0,\infty)} \widetilde{\kappa}_t \, d(\widetilde{Y}_t^2) \text{ subject to } \widetilde{Y} \in \widetilde{\mathscr{Y}},$$

*where* $\widetilde{\kappa}_t \triangleq \widetilde{\lambda}_t/\rho_t$, $\widetilde{\lambda}_t \triangleq \sup_{u\geq t}\lambda_u$, $t \geq 0$, *and*

$$\widetilde{\mathscr{Y}} \triangleq \left\{ (\widetilde{Y}_t)_{t\geq 0} \text{ right-continuous, increasing } : \widetilde{Y}_{0-} \triangleq \eta_0, \int_{[0,\infty)} \widetilde{\lambda}_t \, d\widetilde{Y}_t = x, \widetilde{K}(\widetilde{Y}) < \infty \right\}.$$

*Moreover, any solution $\widetilde{Y}^*$ to (3.5) with $\{d\widetilde{Y}^* > 0\} \subset \{\widetilde{\lambda} = \lambda\}$ will also be a solution to (3.4).*

*Remark* 3.1. For an increasing process $Y = (Y_t)_{t\geq 0}$ we say that $t$ is a point of increase toward the right and write $dY_t > 0$ if $Y_{t-} < Y_u$ for any $u > t$. A similar convention applies to decreasing processes and points of decrease toward the right.

The next proposition describes the (necessary and sufficient) first-order conditions for optimality in problem (3.5). As one would expect, the broker has to strike a balance between the impact of current orders on future mark-up costs (as represented by the left side of (3.6) below) and the current prospect on future market conditions (as represented by the decreasing envelope $\widetilde{\lambda}$ of market depth over resilience on the right side of that equation).

Proposition 3.5. *For $\widetilde{\kappa}$, $\widetilde{\lambda} \geq 0$ as in Theorem 3.4, $\widetilde{Y}^* \in \widetilde{\mathscr{Y}}$ solves (3.5) if and only if there is a constant $y > 0$ such that*

$$\text{(3.6)} \qquad -\int_{[t,\infty)} \widetilde{Y}_u^* \, d\widetilde{\kappa}_u \geq y\widetilde{\lambda}_t \text{ for } t \geq 0 \text{ with “=” whenever } d\widetilde{Y}_t^* > 0\,.$$

Constructing right-continuous increasing $\widetilde{Y}^* \geq 0$ satisfying the first-order conditions of (3.6) can be done by using a time-change and concave envelopes; see also Figure 3 below.

Theorem 3.6. *Under Assumptions 2.1 and 2.2, consider the level passage times $\tau_k \triangleq \inf\{t \geq 0 \,:\, \widetilde{\kappa}_t \leq k\}$, and let $\widetilde{\Lambda}_k \triangleq k\rho_{\tau_k}$, $k \in (0, \widetilde{\kappa}_0]$, and $\widetilde{\Lambda}_0 \triangleq 0$.*

*Then $\widetilde{\Lambda}$ is a continuous increasing map on $[0, \widetilde{\kappa}_0]$. Its concave envelope $\widehat{\Lambda}$ is absolutely continuous with a left-continuous decreasing density $\partial\widehat{\Lambda} = (\partial\widehat{\Lambda}_k)_{0<k\leq\widetilde{\kappa}_0} \geq 0$. Moreover, letting $\partial\widehat{\Lambda}_0 \triangleq \partial\widehat{\Lambda}_{0+}$, we have that for any $y > 0$ and $\eta_0 \geq 0$, $\widetilde{Y}_t^* \triangleq (y\partial\widehat{\Lambda}_{\widetilde{\kappa}_t}) \vee \eta_0$, $t \geq 0$, with $\widetilde{Y}_{0-}^* \triangleq \eta_0$, yields a right-continuous increasing process satisfying (3.6).*

Combining the previous results, we obtain the following solution to our original problem (2.3) which also provides a characterization different from that outlined in Theorem 3.1; see also Figure 2 below.

Corollary 3.7. *Under the assumptions of Theorem 3.6 and using its notation, we have the following dichotomy.*

*In case $|\partial\widehat{\Lambda}|_{\mathbf{L}^2} \triangleq (\int_0^{\widetilde{\kappa}_0} (\partial\widehat{\Lambda}_k)^2 \, dk)^{\frac{1}{2}} < \infty$ we can choose $y^* > 0$ uniquely such that*

$$\text{(3.7)} \qquad X_t^* \triangleq \lambda_0 (y^*\partial\widehat{\Lambda}_{\widetilde{\kappa}_0} - \eta_0)^+ + \int_{(0,t]} \lambda_s \, d\left\{ (y^*\partial\widehat{\Lambda}_{\widetilde{\kappa}_s}) \vee \eta_0 \right\}, \quad t \geq 0,$$

*increases from $X_{0-}^* \triangleq 0$ to $X_\infty^* = x$; this $X^* \in \mathscr{X}$ is an optimal order schedule for problem (2.3). In the special case where $\eta_0 = 0$, we have $y^* = x/|\partial\widehat{\Lambda}|_{\mathbf{L}^2}^2$, and the minimal costs are given by $C(X^*) = x^2/(2|\partial\widehat{\Lambda}|_{\mathbf{L}^2}^2)$.*

*If, by contrast,* $|\partial\widehat{\Lambda}|_{\mathbf{L}^2} = \infty$*, then we have* $\inf_{X\in\mathscr{X}} C(X) = 0$*, and problem* (2.3) *does not have a solution.*

## 4. Illustrations.

Corollary 3.7 reduces the construction of optimal order schedules to the computation of a concave envelope. This can often be done in closed form; see, e.g., our treatment in section 4.1 of the constant parameter case from Obizhaeva and Wang [7]. Alternatively, one can resort to highly efficient numerical methods from discrete geometry to come up with solutions to essentially arbitrary liquidity patterns, as we illustrate in section 4.2.

### 4.1. Constant market depth and resilience.

Let us first show how to recover the solution of Obizhaeva and Wang [7], who consider a time horizon $T > 0$, constant market depth $\delta_t \equiv \delta_0 1_{[0,T]}(t)$, and constant market resilience $r_t \equiv r_0 > 0$, $t \geq 0$. In this case we have

$$\lambda_t = \widetilde{\lambda}_t = \delta_0 e^{-r_0 t} 1_{[0,T]}(t) \text{ and } \kappa_t = \widetilde{\kappa}_t = \delta_0 e^{-2r_0 t} 1_{[0,T]}(t)\,.$$

Hence,

$$\rho_{\tau_k} = \sqrt{\delta_0/(k \vee \kappa_T)} \text{ and } \widetilde{\Lambda}_k = \sqrt{\delta_0 k} \wedge (\sqrt{\delta_0/\kappa_T}k), \quad 0 \leq k \leq \delta_0.$$

Thus, $\widetilde{\Lambda}$ is its own concave envelope, i.e., $\widetilde{\Lambda} = \widehat{\Lambda}$, and its left-continuous density is

$$\partial\widehat{\Lambda}_k = \begin{cases} \frac{1}{2}\sqrt{\delta_0/k}, & k > \kappa_T, \\ \sqrt{\delta_0/\kappa_T} = e^{r_0 T}, & k \leq \kappa_T. \end{cases}$$

Obviously $\partial\widehat{\Lambda}$ is square integrable (and hence the problem is well-posed) if and only if $T < \infty$. In that case, we compute

$$\widehat{Y}_t \triangleq \partial\widehat{\Lambda}_{\widetilde{\kappa}_t} = \begin{cases} \frac{1}{2}\sqrt{\delta_0/\kappa_t} = \frac{1}{2}e^{r_0 t}, & t < T, \\ \sqrt{\delta_0/\kappa_T} = e^{r_0 T}, & t \geq T, \end{cases}$$

and for any $y > 0$ the order schedule from (3.7),

$$X_t^y \triangleq \delta_0 \left(\frac{1}{2}y - \eta_0\right)^+ + \frac{1}{2}y\delta_0 r_0 \left(t \wedge T - \tau^y\right)^+ + \frac{1}{2}\delta_0 y 1_{[T,\infty]}(t), \quad t \geq 0,$$

with

$$\tau^y \triangleq \begin{cases} \left(\frac{1}{r_0}\log\frac{2\eta_0}{y}\right)^+ \wedge T, & y > 2\eta_0 e^{-r_0 T}, \\ T, & \eta_0 e^{-r_0 T} \leq y \leq 2\eta_0 e^{-r_0 T}, \\ \infty, & y < \eta_0 e^{-r_0 T}, \end{cases}$$

is optimal for the total volume it trades. In particular, if $\eta_0 = 0$, we find that

$$X_t^y = \frac{y\delta_0}{2}\left(1 + r_0(t \wedge T) + 1_{[T,\infty]}(t)\right), \quad t \geq 0.$$

So choosing $y^* \triangleq x/(\delta_0(1 + r_0 T/2))$ yields $X^* = X^{y^*}$ with $X^*_\infty = x$. We therefore recover the result of Obizhaeva and Wang [7]: If $\eta_0 = 0$, i.e., if there have been no previous orders, it is optimal to place orders of size $y^*\delta_0/2$ at both $t = 0$ and $t = T$ and to place orders at the constant rate $y^*\delta_0 r_0/2$ in between; cf. Figure 1.

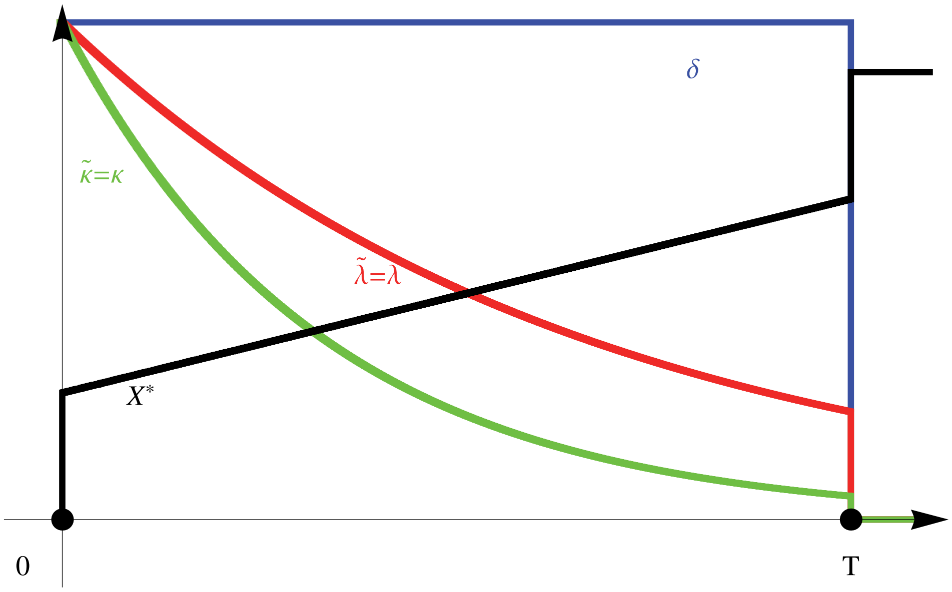


**Figure 1.** *Optimal order schedule $X^*$ (black) for constant market depth $\delta$ (blue), its resilience adjustment $\lambda = \widetilde{\lambda}$ (red), $\kappa = \widetilde{\kappa}$ (green) over a finite horizon $T$.*

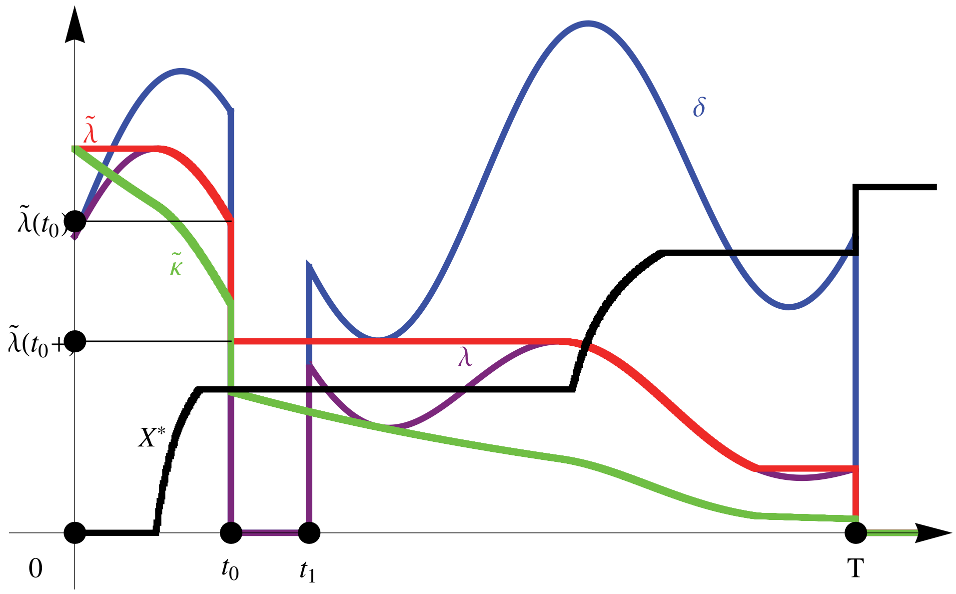


**Figure 2.** *A specification of market depth $\delta$ (blue) with finite horizon $T$, its resilience adjustment $\lambda$ (purple), the corresponding decreasing envelopes $\widetilde{\lambda}$ (red) and $\widetilde{\kappa}$ (green), along with an optimal order schedule $X^*$ (black).*

**4.2. Time-varying market depth.** We next illustrate that the above order placement strategy of [7] is indeed strongly dependent on constant market depth and resilience. Figure 2 exhibits how a fluctuating market depth affects the timing of the optimal order placement as provided by Corollary 3.7. Note that we include a shut-down period for the market over the

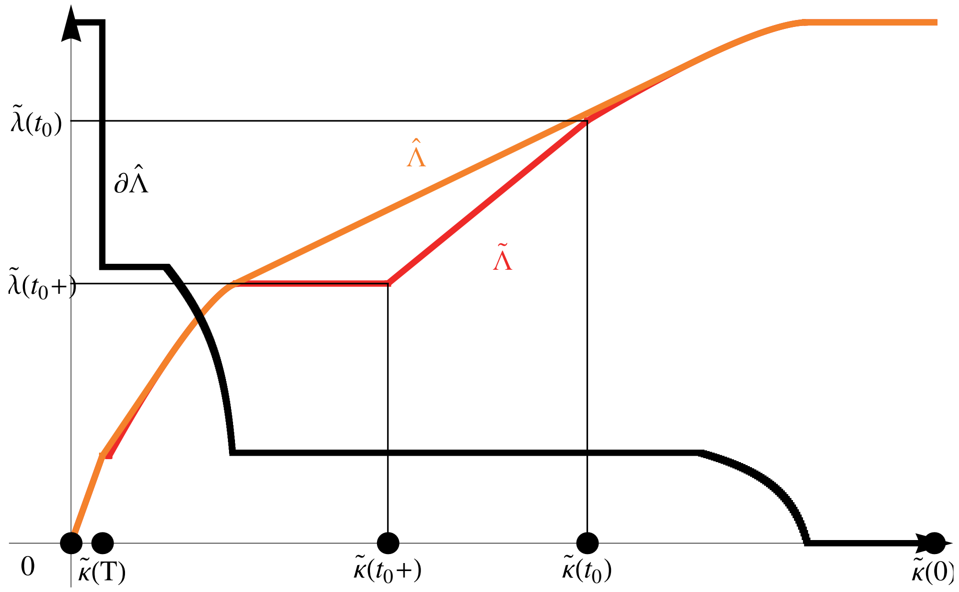


**Figure 3.** *The decreasing envelope of resilience adjusted market depth $\widetilde{\Lambda}$ (red), its concave envelope $\widehat{\Lambda}$ (orange), and the density $\partial\widehat{\Lambda}$ (black).*

time period $(t_0, t_1)$ when market depth vanishes. The corresponding concepts introduced by Theorem 3.6 are illustrated in Figure 3.

If we decrease the resilience parameter to $r_0 = 0$, i.e., we assume permanent price impact of the broker's orders, the focus on peaks of market depth sharpens to the extent that eventually only one huge order is placed when market depth reaches its global maximum; see Figure 4.

Proposition 4.1. *If $r \equiv 0$ and $\delta$ satisfies Assumption 2.2, the solutions to optimization problem (2.3) are precisely those order schedules $X^* \in \mathscr{X}$ with $\{dX^* > 0\} \subset \arg\max \delta$.*

*Proof.* When $r \equiv 0$, $\rho \equiv 1$, and so $\eta_t^X = \eta_0 + \int_{[0,t]} \frac{dX_s}{\delta_s} \geq \eta_0 + \frac{X_t}{\max\delta}$, $t \geq 0$. Thus,

$$C(X) \geq \eta_0 x + \frac{x^2}{2\max\delta}, \quad X \in \mathscr{X},$$

with equality for all $X^* \in \mathscr{X}$ with $\{dX^* > 0\} \subset \arg\max\delta$. ■

Conversely, with high resilience, orders tend to be spread out more around local maxima of market depth, as illustrated by Figure 5. Figures 2 and 5 also show that the precise moments when it is optimal to issue orders would be hard to guess in advance. Hence, an approach via classical calculus of variations as in Fruth, Schöneborn, and Urusov [5] or via the methods of Acevedo and Alfonsi [1] seems infeasible in these general cases.

**5. Proofs.** We first prove that the original problem (2.3) can indeed be reformulated as (3.4) by giving the following proof.

*Proof of Proposition* 3.2. We first observe that for $X \in \mathscr{X}$ the mapping in (3.3) defines an increasing right-continuous $Y$ with $Y = \rho\eta^X$. Because $C(X) < \infty$, $\eta^X$ is $dX$-integrable and thus finite on $\{X < x\}$. Hence, $Y$ is finite on this set as well, and we conclude $dX = \lambda\, dY$. It follows by elementary calculus that $K(Y) = C(X)$ and, thus, $Y \in \mathscr{Y}$, as desired.

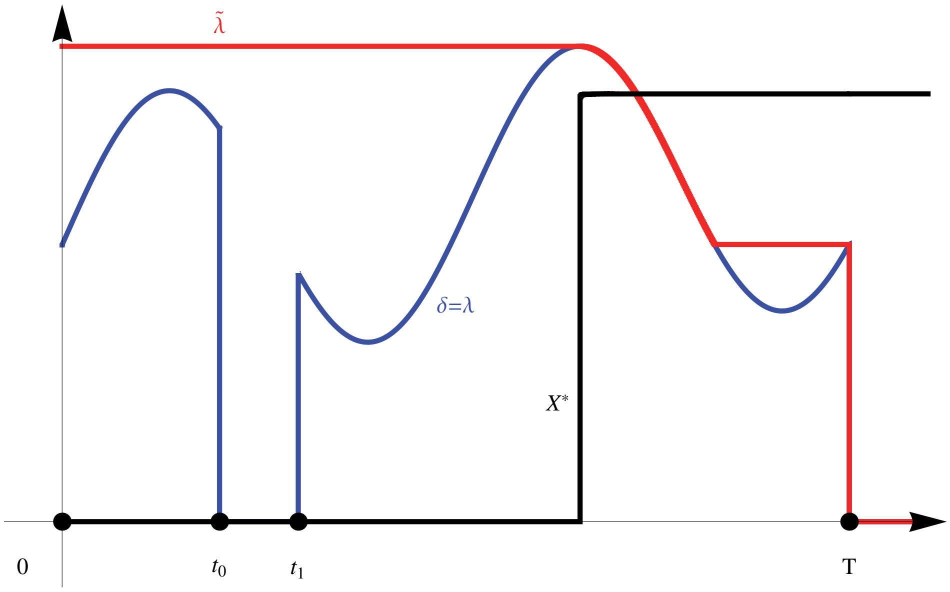


**Figure 4.** *Optimal order schedule $X^*$ (black) without market resilience and time-varying market depth $\delta$ (blue).*

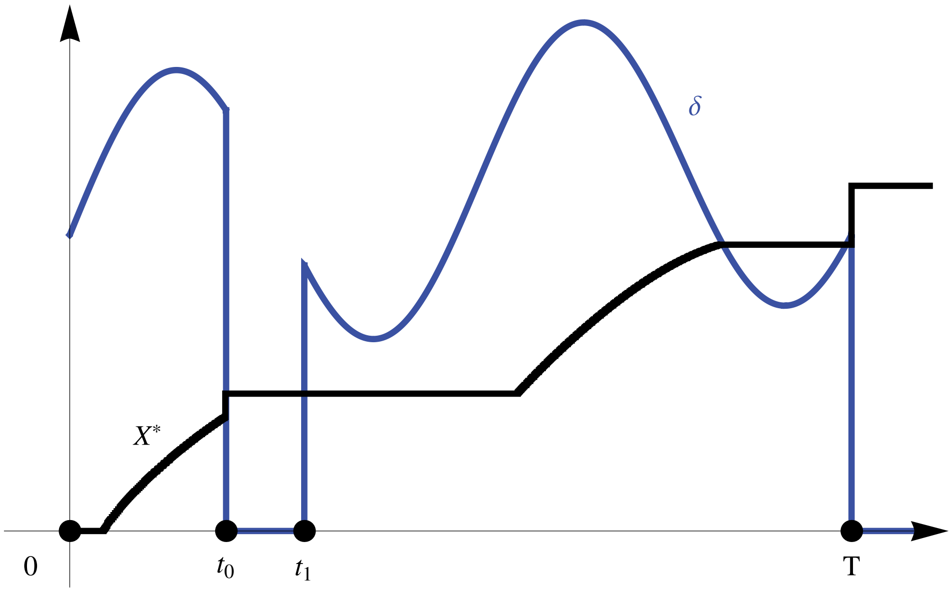


**Figure 5.** *Optimal order schedule $X^*$ (black) with strong market resilience for time-varying market depth $\delta$ (blue).*

Conversely, for $Y \in \mathscr{Y}$, $\kappa = \lambda/\rho$ is $d(Y^2)$-integrable. Since $\rho > 0$ is continuous this implies that $\lambda$ is locally $dY$-integrable, and so $X$ given by (3.3) is right-continuous and increasing with $dX = \lambda\, dY$. By the same reasoning as above this implies $C(X) = K(Y)$ as well as $X \in \mathscr{X}$. ■

We next characterize when problem (3.4) is convex.

*Proof of Proposition* 3.3. If $\kappa$ is upper-semicontinuous and decreasing, it is also left-continuous, and we can use Fubini's theorem to write

$$K(Y) = \frac{1}{2}\left(\kappa_\infty(Y_\infty^2 - \eta_0^2) - \int_{[0,\infty)} (Y_t^2 - \eta_0^2)\, d\kappa_t\right)$$

for any right-continuous increasing $Y$ with $Y_{0-} = \eta_0$. Hence, $K = K(Y)$ is obviously convex in such $Y$ with strict convexity holding true on its domain for strictly decreasing $\kappa$.

Conversely, consider for $0 \leq s < t < \infty$ the function $Y \triangleq \eta_0 + a1_{[s,\infty]} + b1_{[t,\infty]}$. Then

$$\begin{aligned}K(Y) &= \frac{1}{2}\left(\kappa_s((a+\eta_0)^2 - \eta_0^2) + \kappa_t\left((a+b+\eta_0)^2 - (a+\eta_0^2)\right)\right)\\ &= \frac{1}{2}\kappa_s a^2 + \kappa_t ab + \frac{1}{2}\kappa_t b^2 + \eta_0(a\kappa_s + b\kappa_t)\end{aligned}$$

is convex in $a, b > 0$ if and only if $\kappa_s \geq \kappa_t \geq 0$, with strict inequalities corresponding to strict convexity. ■

In order to prepare the proof of Theorem 3.4 let us recall that for any increasing $Z : [0,\infty) \to \mathbf{R}$ we let

$$\{dZ > 0\} \triangleq \{t \geq 0 \ : \ Z_{t-} < Z_u \quad \text{for all} \quad u > t\}$$

denote the collection of all points of increase toward the right. For a decreasing $Z$ we let $\{dZ < 0\} \triangleq \{d(-Z) > 0\}$. In either case we let $\operatorname{supp} dZ$ denote the support of the measure $dZ$, i.e., the smallest closed set whose complement has vanishing $dZ$-measure.

Lemma 5.1. *For upper-semicontinuous bounded $\lambda : [0,\infty) \to \mathbf{R}$, we have that $\widetilde{\lambda}_t \triangleq \sup_{u \geq t} \lambda_u$ is left-continuous and decreasing with*

$$\{d\widetilde{\lambda} < 0\} \subset \{\widetilde{\lambda} = \lambda\}\,. \tag{5.1}$$

*Moreover, we have the partition*

$$\mathbf{R} = \{d\widetilde{\lambda} < 0\} \cup \bigcup_{n \in N_1} [l_n, r_n) \cup \bigcup_{n \in N_2} (l_n, r_n), \tag{5.2}$$

*where $(l_n, r_n)$, $n \in N$, are the disjoint open intervals forming $\mathbf{R}\backslash \operatorname{supp} d\widetilde{\lambda}$ and where $N_1 = \{n \in N \ : \ l_n \geq 0, \Delta_{l_n}\widetilde{\lambda} = 0\}$ and $N_2 = N\backslash N_1$.*

*Proof.* Left-continuity of $\widetilde{\lambda}$ and relation (5.1) are immediate. Note next that $\{d\widetilde{\lambda} < 0\} \subset \operatorname{supp} d\widetilde{\lambda}$ and therefore $\mathbf{R}\backslash\{d\widetilde{\lambda} < 0\} \supset \bigcup_{n\in N}(l_n, r_n)$. Hence, to deduce partition (5.2) it suffices to observe that for $n \in N_1$ we have $l_n \notin \{d\widetilde{\lambda} < 0\}$ and that for $t \geq 0$ such that $\widetilde{\lambda}_t = \widetilde{\lambda}_u$ for some $u > t$ we have $(t, u) \subset (l_n, r_n)$ for some $n \in N$, and thus $t \in (l_n, r_n)$ or $t = l_n$ with $\Delta_{l_n}\widetilde{\lambda} = 0$. ■

The main tool in the proof of Theorem 3.4 is the following lemma.

Lemma 5.2. *Under the conditions of Theorem 3.4, we can find for any increasing right-continuous $Y \geq \eta_0$ an increasing right-continuous $\widetilde{Y} \geq \eta_0$ such that $\widetilde{Y} \leq Y$ and*

(i) $\int_{[0,\infty)} \lambda_t \, dY_t = \int_{[0,\infty)} \lambda_t \, d\widetilde{Y}_t$,

(ii) $\{d\widetilde{Y} > 0\} \subset \{d\widetilde{\lambda} < 0\}$, *and*

(iii) $K(Y) \geq K(\widetilde{Y}) = \widetilde{K}(\widetilde{Y})$.

*Proof.* We let $I_n$, $n \in N$, denote the disjoint intervals of Lemma 5.1 forming the complement of $\{d\widetilde{\lambda} < 0\}$, and we use $l_n$, $r_n$ to denote their respective boundaries. For the one interval $I_n$ whose left bound is $l_n = -\infty$ we now redefine, for simplicity of notation, $l_n \triangleq 0$ provided that $r_n > 0$; if, by contrast, this $I_n$ is just the negative half line we can and shall remove it from consideration in what follows. Similarly, if $r_n = \infty$ for some $n \in N$, it follows from Assumption 2.2 that $\delta_t = \lambda_t = \kappa_t \equiv 0$ on $I_n$, which thus can be disregarded as well.

Observe then that

$$\sup_{I_n} \lambda = \lambda_{r_n} \tag{5.3}$$

by upper-semicontinuity of $\lambda$ and our choice of when and when not to include $l_n$ in $I_n$.

Let, for $t \geq 0$,

$$\widetilde{Y}_t \triangleq \eta_0 + \int_{[0,t]} 1_{\{d\widetilde{\lambda}<0\}}(s) \, dY_s + \sum_{n \in N, r_n \leq t} \int_{I_n} \frac{\lambda_s}{\lambda_{r_n}} \, dY_s.$$

We first note that $\widetilde{Y} \leq Y$. Indeed,

$$\begin{aligned} Y_t - \widetilde{Y}_t &= \int_{[0,t]} 1_{\mathbf{R}\setminus\{d\widetilde{\lambda}<0\}}(s) \, (dY_s - d\widetilde{Y}_s) \\ &= \sum_{n \in N, l_n \leq t} \left( \int_{I_n \cap [0,t]} dY_s - 1_{[r_n,\infty)}(t) \int_{I_n} \frac{\lambda_s}{\lambda_{r_n}} \, dY_s \right) \end{aligned}$$

is nonnegative because of (5.3).

Assertion (i) is readily checked using the partition given by (5.2). For assertion (ii) it suffices to observe that all $r_n$, $n \in N$, are contained in $\{d\widetilde{\lambda} < 0\}$.

In order to prove assertion (iii), we first note that $K(\widetilde{Y}) = \widetilde{K}(\widetilde{Y})$ is an immediate consequence of (ii) and (5.1). To establish $K(Y) - K(\widetilde{Y}) \geq 0$ we decompose this difference into its contributions from the different parts in the partition given by (5.2), each of which will be shown to be nonnegative.

From $\{d\widetilde{\lambda} < 0\} \setminus \{r_n : n \in N\}$ we collect

$$\begin{aligned} &\frac{1}{2} \int_{[0,\infty) \cap (\{d\widetilde{\lambda}<0\} \setminus \{r_n \,:\, n \in N\})} \kappa_t \left[ d(Y_t^2) - d(\widetilde{Y}_t^2) \right] \\ &= \int_{[0,\infty) \cap (\{d\widetilde{\lambda}<0\} \setminus \{r_n \,:\, n \in N\})} \kappa_t \left[ \left( Y_{t-} + \frac{1}{2} \Delta_t Y \right) dY_t - \left( \widetilde{Y}_{t-} + \frac{1}{2} \Delta_t \widetilde{Y} \right) d\widetilde{Y}_t \right], \end{aligned}$$

which is nonnegative because $Y \geq \widetilde{Y}$ and because $dY_t = d\widetilde{Y}_t$ for $t \in \{d\widetilde{\lambda} < 0\} \setminus \{r_n : n \in N\}$ by construction.

From $I_n \cup \{r_n\}$, $n \in N$, we get the contribution

$$\frac{1}{2}\left\{\int_{I_n\cup\{r_n\}} \kappa_t d(Y_t^2) - \kappa_{r_n}\left[\left(\widetilde{Y}_{r_n-} + \int_{I_n\cup\{r_n\}} \frac{\lambda_s}{\lambda_{r_n}}\, dY_s\right)^2 - \widetilde{Y}^2_{r_n-}\right]\right\},$$

for which we note that its $[\dots]$-part can be written as

$$\begin{aligned}
&\frac{1}{2}\left[\left(\widetilde{Y}_{r_n-} + \int_{I_n\cup\{r_n\}} \frac{\lambda_s}{\lambda_{r_n}}\, dY_s\right)^2 - \widetilde{Y}^2_{r_n-}\right]\\
=&\frac{1}{2}\left(\int_{I_n\cup\{r_n\}} \frac{\lambda_s}{\lambda_{r_n}}\, dY_s\right)^2 + \widetilde{Y}_{r_n-}\int_{I_n\cup\{r_n\}} \frac{\lambda_s}{\lambda_{r_n}}\, dY_s\\
=&\int_{I_n\cup\{r_n\}}\int_{(I_n\cup\{r_n\})\cap[l_n,t)} \frac{\lambda_s}{\lambda_{r_n}}\frac{\lambda_t}{\lambda_{r_n}}\, dY_s\, dY_t + \frac{1}{2}\sum_{\Delta_t Y\neq 0, t\in I_n\cup\{r_n\}} \left(\frac{\lambda_t}{\lambda_{r_n}}\right)^2 (\Delta_t Y)^2\\
&+ \widetilde{Y}_{r_n-}\int_{I_n\cup\{r_n\}} \frac{\lambda_s}{\lambda_{r_n}}\, dY_s.
\end{aligned}$$

Hence, using (5.3) again, we obtain with $y_n \triangleq Y_{l_n-}$ if $l_n \in I_n$ and $y_n \triangleq Y_{l_n}$ otherwise that

$$\begin{aligned}
\frac{1}{2}[\dots] \leq& \int_{I_n\cup\{r_n\}} (Y_{t-} - y_n)\frac{\lambda_t}{\lambda_{r_n}}\, dY_t + \frac{1}{2}\sum_{\Delta_t Y\neq 0, t\in I_n\cup\{r_n\}} \frac{\lambda_t}{\lambda_{r_n}}(\Delta_t Y)^2\\
&+ \widetilde{Y}_{r_n-}\int_{I_n\cup\{r_n\}} \frac{\lambda_s}{\lambda_{r_n}}\, dY_s\\
\leq& \int_{I_n\cup\{r_n\}} (Y_{t-} - y_n)\frac{\lambda_t}{\lambda_{r_n}}\, dY_t + \frac{1}{2}\sum_{\Delta_t Y\neq 0, t\in I_n\cup\{r_n\}} \frac{\lambda_t}{\lambda_{r_n}}(\Delta_t Y)^2\\
&+ y_n\int_{I_n\cup\{r_n\}} \frac{\lambda_s}{\lambda_{r_n}}\, dY_s\\
=&\frac{1}{2}\int_{I_n\cup\{r_n\}} \frac{\lambda_t}{\lambda_{r_n}}\, d(Y_t^2),
\end{aligned}$$

where the second estimate holds since $\widetilde{Y}_{r_n-} = \widetilde{Y}_{l_n} \leq y_n$ because of (ii). Since $\rho = \lambda/\kappa$ is increasing by assumption, we have

$$\frac{\lambda_t}{\lambda_{r_n}} = \frac{\rho_t}{\rho_{r_n}}\frac{\kappa_t}{\kappa_{r_n}} \leq \frac{\kappa_t}{\kappa_{r_n}},$$

and thus

$$\frac{1}{2}\kappa_{r_n}[\dots] \leq \frac{1}{2}\int_{I_n\cup\{r_n\}} \kappa_t\, d(Y_t^2),$$

as remained to be shown. ■

With the preceding policy improvement lemma it is now easy to give the following proof.

*Proof of Theorem* 3.4. By Lemma 5.2 and using its notation, we can find for any $Y \in \mathscr{Y}$ a $\widetilde{Y} \in \widetilde{\mathscr{Y}} \cap \mathscr{Y}$ such that

$$\widetilde{K}(Y) \geq K(Y) \geq K(\widetilde{Y}) = \widetilde{K}(\widetilde{Y}).$$

As a result, $\inf_{\mathscr{Y}} K = \inf_{\widetilde{\mathscr{Y}}} \widetilde{K}$. Moreover, if $\widetilde{Y}^* \in \widetilde{\mathscr{Y}}$ attains the latter infimum, we can apply Lemma 5.2 to $\widetilde{\lambda}$ and $\widetilde{K}$ instead of $\lambda$ and $K$ to obtain another optimal $\widetilde{Y}^{**} \in \widetilde{\mathscr{Y}}$ which satisfies in addition $\{d\widetilde{Y}^{**} > 0\} \subset \{d\widetilde{\lambda} < 0\}$. By Lemma 5.1, the latter set is contained in $\{\lambda = \widetilde{\lambda}\} = \{\kappa = \widetilde{\kappa}\}$, and thus this $\widetilde{Y}^{**}$ is also contained in $\mathscr{Y}$ and optimal for (3.4) as well. ■

Let us next derive the first-order conditions of the convexified problem (3.5) in the following proof.

*Proof of Proposition* 3.5. Recalling that $\widetilde{\kappa}_\infty = 0$, we obtain by Fubini's theorem

$$\widetilde{K}(Y) = -\frac{1}{2}\int_{[0,\infty)} (Y_t^2 - \eta_0^2)\, d\widetilde{\kappa}_t. \tag{5.4}$$

For necessity, we observe that for any $Y \in \widetilde{\mathscr{Y}}$ and $0 < \varepsilon \leq 1$ we have

$$\begin{aligned} 0 &\leq \widetilde{K}(\varepsilon Y + (1-\varepsilon)Y^*) - \widetilde{K}(Y^*) \\ &= -\varepsilon \int_{[0,\infty)} (Y_t - Y_t^*)Y_t^*\, d\widetilde{\kappa}_t - \frac{\varepsilon^2}{2}\int_{[0,\infty)} (Y_t - Y_t^*)^2\, d\widetilde{\kappa}_t, \end{aligned}$$

which, upon division by $\varepsilon > 0$ and letting $\varepsilon \downarrow 0$, yields that $Y^*$ also solves the linear problem

$$\text{Minimize} \quad -\int_{[0,\infty)} Y_t^* Y_t\, d\widetilde{\kappa}_t \text{ subject to } Y \in \widetilde{\mathscr{Y}}. \tag{5.5}$$

Equivalently, due to Fubini's theorem, $Y^*$ is a solution to the problem

$$\text{Minimize} \quad \int_{[0,\infty)} \left(-\int_{[t,\infty)} Y_u^*\, d\widetilde{\kappa}_u\right) dY_t \text{ subject to } Y \in \widetilde{\mathscr{Y}}. \tag{5.6}$$

As a consequence, $Y^*$ can solve (5.5) only if $dY_t^* > 0$ exclusively at those times $t \geq 0$ when $-\int_{[t,\infty)} Y_u^*\, d\widetilde{\kappa}_u/\widetilde{\lambda}_t$ attains its infimum over $\{\widetilde{\lambda} > 0\}$. Hence, this infimum is actually a minimum and is thus strictly positive. Denoting it by $y > 0$ shows the necessity of (3.6).

For sufficiency we use (5.4) again to deduce that for $Y \in \widetilde{\mathscr{Y}}$,

$$\widetilde{K}(Y) - \widetilde{K}(Y^*) = -\frac{1}{2}\int_{[0,\infty)} ((Y_t)^2 - (Y_t^*)^2)\, d\widetilde{\kappa}_t \geq -\int_{[0,\infty)} Y_t^*(Y_t - Y_t^*)\, d\widetilde{\kappa}_t.$$

The last term is nonnegative if $Y^*$ solves (5.5), which due to the equivalence of (5.5) and (5.6) amounts to our first-order condition (3.6). ■

The construction of solutions to the first-order conditions given in Theorem 3.6 can now be established.

*Proof of Theorem* 3.6. $\widetilde{\Lambda}$ is continuous on $[0,\widetilde{\kappa}_0]$ since $k \mapsto \tau_k$ is continuous because of the strict monotonicity of $\rho$ and, thus, of $\widetilde{\kappa}$ on $\{\widetilde{\kappa} > 0\}$. $\widetilde{\Lambda}$ is increasing because, along with $\widetilde{\kappa}_t$, also $\widetilde{\Lambda}_{\widetilde{\kappa}_t} = \widetilde{\kappa}_t \rho_t = \widetilde{\lambda}_t$ is decreasing in $t \geq 0$. Absolute continuity of the concave envelope $\widehat{\Lambda}$ follows from the continuity of $\widetilde{\Lambda}$.

The monotonicity of $\widetilde{Y}^*$ is obvious from the monotonicity of $\widetilde{\kappa}$ and $\partial\widehat{\Lambda}$. For its right-continuity note that $\lim_{t\downarrow t_0} \widetilde{Y}^*_t = (y\partial\widehat{\Lambda}_{\widetilde{\kappa}_{t_0+}}) \vee \eta_0$ by left-continuity of $\partial\widehat{\Lambda}$ and its definition at 0. Hence, our assertion amounts to $\partial\widehat{\Lambda}_{k_0} = \partial\widehat{\Lambda}_{k_1}$, where $k_0 \triangleq \widetilde{\kappa}_{t_0+}$ and $k_1 \triangleq \widetilde{\kappa}_{t_0} \geq k_0$. If $k_0 = k_1$, there is nothing to show. In case $k_0 < k_1$, $\tau_k = \tau_{k_1}$ for $k \in [k_0, k_1)$, and, thus, $\widetilde{\Lambda}$ is linear with slope $\rho_{\tau_{k_0}}$ on this interval. As a consequence, $\widehat{\Lambda}$ is linear there as well, and, thus, $\partial\widehat{\Lambda}_{k_1} = \partial\widehat{\Lambda}_{k_0+}$ by left-continuity of $\partial\widehat{\Lambda}$. Hence, it suffices to show that there is no downward jump in $\partial\widehat{\Lambda}$ at $k_0$. If there were such a jump, then, by the properties of concave envelopes, necessarily $\widehat{\Lambda}_{k_0} = \widetilde{\Lambda}_{k_0}$ and $\partial\widehat{\Lambda}_{k_0+} \geq \rho_{\tau_{k_0}}$. Hence, for $k \leq k_0$ we would have

$$k\rho_{\tau_{k_0}} \leq \widetilde{\Lambda}_k \leq \widehat{\Lambda}_k \leq \widehat{\Lambda}_{k_0} + \partial\widehat{\Lambda}_{k_0+}(k - k_0) \leq k\rho_{\tau_{k_0}},$$

where the first estimate is due to the monotonicity of $\rho$, the second is the envelope property of $\widehat{\Lambda}$, the third follows from its concavity, and the last is a consequence of the just derived properties of $\widehat{\Lambda}$ and $\partial\widehat{\Lambda}$ at $k_0$. We would thus have equality everywhere in the above estimates and in particular $\partial\widehat{\Lambda}_{k_0} = \rho_{\tau_{k_0}} \leq \partial\widehat{\Lambda}_{k_0+}$. This is a contradiction to the presumed downward jump of $\partial\widehat{\Lambda}$ at $k_0$.

To verify that $\widetilde{Y}^*$ satisfies the first-order condition (3.6), let us first argue that

$$\begin{aligned} -\int_{[t,\infty)} \widetilde{Y}^*_u \, d\widetilde{\kappa}_u &\geq -y \int_{[t,\infty)} \partial\widehat{\Lambda}_{\widetilde{\kappa}_u} \, d\widetilde{\kappa}_u = y \int_0^{\widetilde{\kappa}_t} \partial\widehat{\Lambda}_k \, dk = y\widehat{\Lambda}_{\widetilde{\kappa}_t} \\ &\geq y\widetilde{\Lambda}_{\widetilde{\kappa}_t} = y\widetilde{\kappa}_t \rho_t = y\widetilde{\lambda}_t. \end{aligned}$$

Indeed, the first estimate is immediate from the definition of $\widetilde{Y}^*$. The first identity follows by the change-of-time formula for Lebesgue–Stieltjes integrals: just observe that $\widetilde{\kappa}_\infty = 0$ by Assumption 2.2 and that $\partial\widehat{\Lambda}$ is constant on those intervals contained in $[0,\widetilde{\kappa}_0]$ which $\widetilde{\kappa}$ jumps across because $\widetilde{\Lambda}$ is linear on such intervals. The second identity follows from the absolute continuity of $\widehat{\Lambda}$ and because $\widehat{\Lambda}_0 = \widetilde{\Lambda}_0 = 0$, again by Assumption 2.2. The second estimate holds because $\widehat{\Lambda} \geq \widetilde{\Lambda}$ by definition of concave envelopes, and for the last identity we note that $\tau_{\widetilde{\kappa}_t} = t$ if $\widetilde{\kappa}_t > 0$ and $\widetilde{\kappa}_t = \widetilde{\lambda}_t = 0$ otherwise. Finally, we observe that $d\widetilde{Y}^*_t > 0$ can only happen when $y\partial\widehat{\Lambda}_{\widetilde{\kappa}}$ has increased above $\eta_0$, which ensures equality in the first of the above estimates. Equality in the second holds for such $t$ as well because if $\partial\widehat{\Lambda}_{\widetilde{\kappa}}$ increases at time $t$, $\partial\widehat{\Lambda}$ must decrease at $\widetilde{\kappa}_t$, and so $\widetilde{\Lambda}$ coincides with its concave envelope $\widehat{\Lambda}$ at this point. ■

We are now in a position to wrap up and give the following proof.

*Proof of Corollary* 3.7. Let $\widehat{Y}_t \triangleq \partial\widehat{\Lambda}_{\widetilde{\kappa}_t}$ and $\widehat{Y}_{0-} \triangleq 0$, and define $Y^y_t \triangleq (y\widehat{Y}_t) \vee \eta_0$, $t \geq 0$, $Y^y_{0-} \triangleq \eta_0$.

As a first step we check that

$$d\widehat{Y}_{t_0} > 0 \text{ only at times } t_0 \geq 0 \text{ when } \widetilde{\lambda}_{t_0} = \lambda_{t_0}. \tag{5.7}$$

In fact, we will show that $d\widetilde{\lambda}_{t_0} < 0$ for such $t_0$. If $\Delta_{t_0}\widetilde{\kappa} < 0$, this is obvious. So let us suppose that $\widetilde{\kappa}_{t_0+} = \widetilde{\kappa}_{t_0}$, and assume that there is $t_1 > t_0$ such that $\widetilde{\lambda}_t = \widetilde{\lambda}_{t_0}$ for $t \in [t_0, t_1]$. In that case, $\widetilde{\Lambda}$ is constant on the interval $(\widetilde{\kappa}_{t_1}, \widetilde{\kappa}_{t_0}]$. Because $d\widehat{Y}_{t_0} > 0$, the density $\partial\widehat{\Lambda}$ must decrease at $k_0 \triangleq \widetilde{\kappa}_{t_0+} = \widetilde{\kappa}_{t_0}$, and so the envelope $\widehat{\Lambda}$ coincides with $\widetilde{\Lambda}$ at this point. Concavity and monotonicity of $\widehat{\Lambda}$ then imply, however, that $\partial\widehat{\Lambda} = 0$ around $k_0$, which is a contradiction to its decrease there.

Let us next prove that $|\partial\widehat{\Lambda}_k|_{\mathbf{L}^2} < \infty$ if and only if $\widetilde{\lambda}$ is $d\widehat{Y}$-integrable. To see this we argue that with $\partial\widehat{\Lambda}_{\widetilde{\kappa}_{0-}} \triangleq 0$ we have

$$\int_{[0,\infty)} \lambda_t \, d\widehat{Y}_t = \int_{[0,\infty)} \widetilde{\Lambda}_{\widetilde{\kappa}_t} \, d(\partial\widehat{\Lambda}_{\widetilde{\kappa}_t}) = \int_{[0,\infty)} \widehat{\Lambda}_{\widetilde{\kappa}_t} \, d(\partial\widehat{\Lambda}_{\widetilde{\kappa}_t})$$

$$= \int_0^{\widetilde{\kappa}_0} \partial\widehat{\Lambda}_l \partial\widehat{\Lambda}_{\widetilde{\kappa}_{\tau_l}} \, dl = \int_0^{\widetilde{\kappa}_0} (\partial\widehat{\Lambda}_l)^2 \, dl.$$

Indeed, the first identity is just (5.7) and the definition of $\widehat{Y}$ and $\widetilde{\Lambda}$. The second identity holds because $\widetilde{\Lambda} = \widehat{\Lambda}$ at points where $\partial\widehat{\Lambda}$ changes; the third identity follows from an application of Fubini's theorem after writing $\widehat{\Lambda}_{\widetilde{\kappa}_t} = \int_0^{\widetilde{\kappa}_t} \partial\widehat{\Lambda}_l \, dl$, and the last equality holds since $\partial\widehat{\Lambda}$ is left-continuous and constant over intervals that $\widetilde{\kappa}$ jumps across.

So if $|\partial\widehat{\Lambda}|_{\mathbf{L}^2} < \infty$, then $X^y_t \triangleq \int_{[0,t]} \lambda \, dY^y$, $t \geq 0$, is real-valued, right-continuous, and increasing in $t$. Moreover, $X^y_\infty$ is increasing in $y \geq 0$ with $X^0_\infty = 0$ and $X^y_\infty \geq X^y_0 \to \infty$ as $y \uparrow \infty$. In fact, $X^y_\infty = y \int_{[0,\infty)} \lambda \, d\widehat{Y}$ for $y \geq \eta_0/\widehat{Y}_0$ (where $0/0 \triangleq \infty$) and, for $y \in [\eta_0/\widehat{Y}_\infty, \eta_0/\widehat{Y}_0]$, $X^y_\infty = y \int_{[\tau_y,\infty)} \lambda \, d\widehat{Y} = \int_{[\tau_{y+},\infty)} \lambda \, d\widehat{Y}$, where $\tau_y \triangleq \inf\{t \geq 0 \; : \; y > \eta_0/\widehat{Y}_t\}$. Hence, $X^y_\infty$ is in fact continuous and strictly increasing from 0 to $\infty$ in $y \geq \eta_0/\widehat{Y}_\infty$, and we thus obtain existence and uniqueness of $y^* > 0$ with $X^{y^*}_\infty = x$. Hence, we can conclude that $X^* \triangleq X^{y^*}$ is contained in $\mathscr{X}$ (and that thus the corresponding $Y^* = Y^{y^*}$ of (3.3) is contained in $\mathscr{Y}$) once we have established that $K(Y^*) < \infty$. For this it suffices to observe that $K(Y^*) \leq (y^*)^2 K(\widehat{Y})$ and that by the same arguments as in our previous calculation of $\int_{[0,\infty)} \lambda \, d\widehat{Y}$ we have

$$K(\widehat{Y}) = \widetilde{K}(\widehat{Y}) = \frac{1}{2} \int_{[0,\infty)} \widetilde{\kappa}_t \, d\left( (\partial\widehat{\Lambda}_{\widetilde{\kappa}_t})^2 \right) = \frac{1}{2} \int_0^{\widetilde{\kappa}_0} (\partial\widehat{\Lambda}_l)^2 \, dl < \infty.$$

We next show that $X^*$ and $Y^*$ are optimal, respectively, for problem (2.3) and problems (3.4) and (3.5). In fact, due to Theorem 3.6, $Y^* = (y^* \partial\widehat{\Lambda}_{\widetilde{\kappa}}) \vee \eta_0$ satisfies the first-order condition (3.6) and, by Proposition 3.5, is thus optimal for the convexified problem (3.5) provided that $Y^*$ is also contained in $\widetilde{\mathscr{Y}}$. To see that even $\int_{[0,\infty)} \widetilde{\lambda} \, dY^* = x$ and to deduce the optimality of $Y^*$ also for problem (3.4) (and thus, by Proposition 3.2, the optimality of $X^*$ for the original problem (2.3)) it suffices by Theorem 3.4 to check $\{dY^* > 0\} \subset \{\lambda = \widetilde{\lambda}\}$, which, in fact, is immediate from (5.7).

The formula for the minimal costs when $\eta_0 = 0$ is an immediate consequence of our above computations for $\widehat{Y}$. It thus remains to show that our optimization problems do not have a solution if $|\partial\widehat{\Lambda}|_{\mathbf{L}^2} = \infty$. To see this note that in this case there is, for any sufficiently large $0 \leq S < T < \infty$, a schedule $X^{S,T} \in \mathscr{X}$ which is optimal for $\delta^{S,T} \triangleq \delta 1_{[S,T]}$ instead of $\delta$ when $\eta_0 = 0$. This follows from our earlier results once we note that the corresponding concave envelope $\widehat{\Lambda}^{S,T}$ always has a bounded density because $T < \infty$, and thus a solution to this finite time horizon problem exists provided its market depth does not vanish identically. This latter condition clearly holds for $\delta^{S,T}$ when $T$ is chosen sufficiently large, for otherwise $\delta \equiv 0$ after some time $S$, which would rule out the presumed explosion of $\partial\widehat{\Lambda}$ at 0. Note that we can furthermore choose $S, T \uparrow \infty$ such that $\widehat{\Lambda}_{\widetilde{\kappa}}$ coincides with $\widetilde{\Lambda}_{\widetilde{\kappa}}$ at these points. This ensures that $\widehat{\Lambda} = \widehat{\Lambda}^{S,T}$ on $[\widetilde{\kappa}_T, \widetilde{\kappa}_S]$ and hence $|\partial\widehat{\Lambda}^{S,T}|^2_{\mathbf{L}^2} \to \int_0^{\widetilde{\kappa}_0} (\partial\widehat{\Lambda}_{k\wedge\widetilde{\kappa}_S})^2\, dk = \infty$ as $T \uparrow \infty$.

Now because

$$C(X^{S,T}) = \int_{[0,\infty)} \frac{\eta_0}{\rho_t}\, dX_t^{S,T} + C^0(X^{S,T}) \le \frac{\eta_0 x}{\rho_S} + C^0(X^{S,T}),$$

where $C^0(X)$ denotes the cost of any $X \in \mathscr{X}$ when $\eta_0 = 0$, we obtain

$$\inf_{\mathscr{X}} C \le \frac{\eta_0 x}{\rho_S} + \frac{x^2}{2|\partial\widehat{\Lambda}^{S,T}|^2_{\mathbf{L}^2}},$$

where we used our formula for the optimal costs $C^0(X^{S,T})$. Because of our special choice of $S, T$, the second term vanishes for any fixed $S$ as $T \uparrow \infty$. The first term vanishes for $S \uparrow \infty$ because $\rho$ has to be unbounded for $\partial\widehat{\Lambda}_k$ to increase to $\infty$ as $k \downarrow 0$. Indeed, $\partial\widehat{\Lambda}_{0+} = \sup_{k>0} \widetilde{\Lambda}_k / k = \sup_{k>0} \rho_{\tau_k}$. ■

Finally, let us show how Theorem 3.1 follows from Corollary 3.7.

*Proof of Theorem* 3.1. In view of Corollary 3.7 it suffices to show that $\sup_{0\le t\le s} L_t^* = \partial\widehat{\Lambda}_{\widetilde{\kappa}_s}$, $s \ge 0$. Now, from the properties of concave envelopes and because of the left-continuity of $\partial\widehat{\Lambda}$ we have for any $0 < k \le \widetilde{\kappa}_0$ that

$$\partial\widehat{\Lambda}_k = \sup_{l\in[k,\widetilde{\kappa}_0]} \inf_{m\in[0,l)} \frac{\widetilde{\Lambda}_m - \widetilde{\Lambda}_l}{m - l}.$$

With the changes of variables $k = \widetilde{\kappa}_s$, $l = \widetilde{\kappa}_t$, $m = \widetilde{\kappa}_u$, the preceding ratio turns into the one occurring in (3.1), accomplishing our proof. ■

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