

OPTIMAL ORDER EXECUTION WITH STOCHASTIC LIQUIDITY

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Von der Fakultät II – Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
– Dr. rer. nat. –

genehmigte Dissertation

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Tag der wissenschaftlichen Aussprache: 14.07.2011

Berlin 2011
D 83

Zusammenfassung

In klassischen Finanzmarktmodellen wird davon ausgegangen, dass Preise nicht davon abhängen, wie viel gehandelt wird. In Wirklichkeit sind Märkte jedoch illiquide, so dass die eigene Handelsstrategie den Preis nachteilig beeinflusst. In der vorliegenden Arbeit wird dieser Preiseinfluss durch ein Modell eines Orderbuchs einer elektronischen Börsenplattform beschrieben. Unter Verwendung dieses Modells betrachten wir das Problem eines institutionellen Investors, der eine große Aktienposition in vorgegebener Zeit kaufen möchte. Gesucht ist die optimale Zerlegung der Order, so dass die gesamten erwarteten Preiseinflusskosten minimiert werden. Wir formulieren diese Fragestellung des Investors als singuläres Kontrollproblem mit drei Zustandsvariablen. Verglichen zu vorhandener Literatur liegt unser Hauptaugenmerk auf der sich zeitlich ändernden Liquidität im Orderbuch. Dies erlaubt uns zu beschreiben, wie der Investor sich in Zeiten relativ hoher bzw. niedriger Liquidität verhalten sollte.

Zunächst behandeln wir den deterministischen Fall, wo wir das Liquiditätsprofil am Anfang des Zeithorizonts fixieren. Wie erwartet lässt sich der Zustandsraum in eine Kauf- und Warteregion zerlegen. In diskreter Zeit können wir per Induktion nachweisen, dass die Struktur dieser Regionen besonders intuitiv ist. In stetiger Zeit lässt sich die Existenz optimaler Strategien zeigen und somit unser Resultat aus diskreter zu stetiger Zeit überführen. In einigen Situationen können wir schließlich explizite Lösungen unseres Optimierungsproblems angeben.

Im Anschluss betrachten wir den Fall stochastischer Liquidität, so dass optimale Strategien sich der Liquiditätsentwicklung anpassen. Es stellt sich als schwierig heraus, dass unsere Kostenfunktion nicht in allen Fällen konvex in der Strategie des Investors ist. Sobald wir diese Konvexität erzwingen, folgt die Eindeutigkeit optimaler Strategien unmittelbar. Gleichzeitig können wir aber auch die Existenz optimaler Strategien zeigen und wiederum das gewünschte Strukturresultat für die Kauf- und Warteregion sicherstellen. Darüber hinaus lassen sich nicht konvexe Fälle stochastischer Liquidität angeben, die das Strukturresultat verletzen.

Zu guter Letzt leiten wir durch Näherung der Zustandsvariablen durch kontrollierte Markovketten ein numerisches Schema her und beweisen dessen Konvergenz. Auf diese Weise können wir die Wertfunktion und die zugehörigen optimalen Strategien näherungsweise berechnen.

Abstract

Classical models in mathematical finance assume that an arbitrary amount of assets can be traded at the current market price. But in reality, markets are illiquid such that trading does have an adverse price impact. In this thesis, this price dependence on trading strategies is described by a model of a limit order book which is relevant in exchange electronic trading systems. Using this model, we consider a large investor who wants to purchase a given amount of shares over a fixed interval of time. We look for the optimal trading schedule such that the total expected costs due to the adverse price impact are minimized. We phrase this optimal execution task of the large investor as a singular control problem with three state dimensions. Compared to the existing literature, our focus is on time-varying liquidity in the limit order book. This allows us to derive how the large investor should trade in periods of comparatively high or low liquidity.

We first treat the deterministic case, where we fix the liquidity profile at the initial time. As one would expect, the state space separates into a no-trading and a trading region. In discrete time, the structure of these regions is found to be particularly intuitive. Together with the fact that we can prove the existence of optimal strategies in continuous time, we can transfer our results from discrete to continuous time. We derive closed-form solution under appropriate conditions.

We go ahead by considering the stochastic liquidity case, where optimal trading strategies react to the liquidity available in the market. A major difficulty is that our cost function may not be convex in the strategies. Enforcing this convexity, uniqueness follows immediately, but we are additionally able to conclude the existence of optimal strategies and again derive convenient structural results concerning the no-trading and trading region. We also construct non-convex stochastic liquidity cases where these structural results fail.

Finally, we establish a convergent numerical scheme which allows us to compute the value function and optimal strategies by approximating the state space variables by a controlled Markov chain.

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Introduction

Economic background

Classical models in mathematical finance assume frictionless markets, and prices do not depend on the trading strategies of market participants. This is a good approach in case of long-term considerations. However, on the time scale of a few trading days or less, it becomes important to incorporate aspects of market microstructure. Due to limited liquidity of real financial markets, trading large volumes moves prices, typically in an unfavorable direction. See for example Harris (2003), Section 2.5, for an illustrative trading story about this issue. The difference between the realized price and the price before the trade is called price impact (or market impact).

In this thesis, we adapt the model of Obizhaeva and Wang (2006) and consider an exogenous impact model with market resilience *and* stochastic liquidity. The model is used to analyze the optimal execution problem of a large investor. To make this more precise, let us give a short overview of the research within the field of market microstructure, which forms the background for our considerations.

Market microstructure

In empirical market microstructure research, data of financial markets are used in order to explain price formation, see e.g. Kraus and Stoll (1972) and Hasbrouck (1991). After the introduction of electronic trading, limit order book data has extensively been used for the same purpose, see Biais, Hillion, and Spatt (1995) and Potters and Bouchaud (2003). During the last decade, hardware improvements and the development of high-speed communication have led to the acceleration of exchange trading such that nowadays, the time between order executions is of the order of a few milliseconds. For such high-frequency data, a new field of literature including Ait-Sahalia and Yu (2008) establishes appropriate statistical and econometric tools. These results are needed for the validation and calibration of the mathematical market microstructure models and execution strategies which we will develop in this thesis.

Non-empirical research

Within the non-empirical market microstructure research, there is quite a rich literature dealing with price impact modeling. The first type of these market microstructure

models can be called endogenous. Their focus is on explaining and deriving price impact. To do so, they take into account the interaction of all market participants such as market makers as well as informed and uninformed agents. Information and inventory play a key role. Prominent examples include the model of Kyle (1985), where the impact turns out to be permanent and fully affects all consequent trades, and Easley and O'Hara (1987), where the impact partly recovers over time. Hence, the endogenous impact models explain the dynamics behind price impact formation, and they can be seen as a support for the exogenous models that we want to concentrate on in the sequel.

Exogenous impact models

A second approach in the field of price impact modeling deals with exogenous models. The dependence of the price impact on the trading strategy and other parameters are fixed at the very beginning and then used in order to find the optimal strategy of a large investor. Gokay, Roch, and Soner (2010) have recently written a survey on this field of exogenous impact models. Papers dealing with the hedging problem of a large investor include Frey (1998), Cetin, Jarrow, Protter, and Yildirim (2004), Bank and Baum (2004) and Rogers and Singh (2010). In the context of hedging, one typically thinks of maturities of a few months. By contrast, this thesis is concerned with the large investor's optimal execution problem, where the time horizon is of the order of days or minutes.

Optimal execution of a large investor

An overview of price impact models that are studied in the optimal execution literature is given by Gatheral (2010). The optimal execution problem has a typical real-life application, where an investor wants to build up (analogously liquidate) an asset position over a fixed period of time. An example could be the case of a bank that borrowed shares to short an asset and now needs to cover its short position in order to return these shares to its broker at the end of the current trading day. Thus, a typical task would be to buy, for instance, five percent of the average daily volume of a stock within the next two hours. In this case, it would not be advisable to trade the entire order in one gulp at the initial time, since this high liquidity demand would result in a tremendous price increase. Due to price recovery effects called resilience, these price impact costs can be decreased considerably by spreading out the order over a longer time interval. This leads to a challenging optimization problem, where one has to find the trade-off between exploiting the price recovery and the urgency to buy due to the fixed time horizon.

The mathematical consideration of such an optimal execution problem is a relatively young field. Bertsimas and Lo (1998) are the first to minimize the total impact costs in the optimal execution problem in a discrete time, linear impact model. Huberman and Stanzl (2004) combine both permanent impact, corresponding to zero resilience, and impact with instantaneous recovery. They show that the permanent impact has to be linear to rule out arbitrage. However, the optimal trade allocation does not only depend on the chosen impact model, but also on the risk criterion of the investor. For

instance, the investor has to be concerned about the trade-off between the prevention of price impact and the risk of the deviation of the fundamental price from the initial asset price due to external influences. First suggestions into this direction using a model with infinite resilience, i.e. instantaneous recovery, have been made by Almgren and Chriss (2001) and Almgren (2003). A mean-variance cost criterion is introduced. Due to its tractability, this model by Almgren is popular in practice. With the same model, Schied and Schöneborn (2009) go further and replace the mean-variance criterion by a general utility function of the large investor. In the sequel, we consider extensions to models that in particular have finite resilience such that for mathematical tractability, we assume a risk-neutral investor, who only takes into account the expected price impact costs instead of its expectation and variance.

The Obizhaeva and Wang model

Like Almgren and Chriss (2001) and the corresponding follow-up papers, the price impact suggested by Obizhaeva and Wang (2006) belongs to the exogenous models and is used to discuss the optimal execution problem. Beyond that, it has some flavor of an endogenous model, since the considered price impact is derived from a simplified limit order book model. This approach allows a more transparent specification of a formula for the impact. Moreover, the resulting optimal execution strategies can be illustrated in the limit order book framework. In particular, the Obizhaeva and Wang (2006) model incorporates the empirically well established resilience effect; see, e.g., Biais, Hillion, and Spatt (1995), Potters and Bouchaud (2003), Bouchaud, Gefen, Potters, and Wyart (2004), and Weber and Rosenow (2005). Price impact resulting from a large trade does not recover immediately, but gradually as liquidity providers position fresh limit orders in the book. For more literature on limit order books, we refer to Cont, Stoikov, and Talreja (2010) and the references therein. Today the predominant amount of trading is done electronically and in doing so, is channeled through limit order books.

Several authors considered an extended version of the Obizhaeva and Wang (2006) impact model. For example Alfonsi, Fruth, and Schied (2010), Predoiu, Shaikhet, and Shreve (2011) and Gatheral, Schied, and Slynko (2010) allow for a price impact which is non-linear in the number of shares traded. The work of Alfonsi, Schied, and Slynko (2009), Gatheral, Schied, and Slynko (2009) and Weiss (2010) focuses on a more general resilience than in the original model.

The main goal of this thesis is to extend the Obizhaeva and Wang (2006) model by introducing stochastic liquidity effects and to analyze the consequences of this kind of liquidity risk. In the following, let us explain this in more detail.

Economic contribution

All of the above mentioned extensions of Obizhaeva and Wang (2006) stick to the assumption that the order book shape stays constant over time and that the resilience is deterministic. In this thesis, we take these two liquidity features to be stochastic. A significant difference between order book depth and resilience is the information available to traders: While the order book depth is visible in many limit order book markets, the resilience is not directly observable and needs to be inferred over time from the evolution of market prices. Due to these informational differences, we deal with the random depths analysis most of the time. But some basic ideas on the stochastic resilience case are also given at the end of this thesis.

The main focus of this thesis is on models with stochastic depth and deterministic resilience. The order book shape depends on the activity of all market participants and should therefore be modeled using a stochastic process instead of being constant over time. As a first step towards this stochastic depth analysis, we address the case of deterministically varying depth. This is interesting in its own right, since it is, e.g., a well-established empirical fact that market activity is U-shaped, i.e., there is less activity in the middle of a trading day than at the beginning and end. See for instance Lorenz and Osterrieder (2009). We go ahead by dealing with the stochastic case. To the best of our knowledge, similar attempts in different settings have only been made by Esser and Mönch (2003), Walia (2006), Roch (2009) and Almgren (2009).

We discuss the optimal execution problem with stochastic depth both in discrete and continuous trading time. From a qualitative point of view, we get the following economic result: Depending on the applied stochastic liquidity process, optimal strategies can both be *aggressive or passive in the liquidity*. By aggressive, as opposed to passive in the liquidity, we mean that the large investor trades intensively, when there is a lot of liquidity available in the market, and suspends trading, when liquidity is scarce. Astonishingly, it is not always optimal to trade aggressive in the liquidity. We get aggressive in the liquidity trading, when the liquidity process is mean-reverting. Without mean-reversion, there is no characteristic level that one could use to judge the current liquidity in the market such that the aggressive in the liquidity behavior might not hold. Therefore it is essential to use reasonable depth dynamics as a model ingredient in order to get reasonable strategies as an output of the optimization procedure. From a quantitative point of view, we can in some deterministic cases explicitly calculate the optimal strategy. For all other cases, we develop rigorous numerical procedures to calculate optimal strategies.

In total, we find a price impact model, incorporating stochastic limit order book depth dynamics, that can be used in algorithmic trading. The calibration can be realized by using empirically observable limit order book data. For the calibration of resilience see Large (2007). We can describe and numerically calculate the trading strategy and minimal expected costs of the large investor in the optimal execution problem. Compared to the deterministic optimal strategy in the basic model with constant liquidity

and essentially constant trading speed, our execution trajectories respond optimally to the stochastically varying depth in the market. Hence, the scenario dependent trajectories can, e.g., have flat stretches where it is optimal not to trade at all. They can considerably outrun or fall behind the linear trajectory of the constant liquidity case.

In the last part of this thesis, we consider constant depth and the resilience is assumed to be either high or low with a given a priori probability distribution. By observing the price evolution after the execution of a market order, the estimate of resilience can be updated over time: If market prices recover quickly, then resilience is likely to be high, whereas a slow market price recovery indicates a low resilience. However, market price evolution is not purely driven by resilience, but also by random price changes. Thus, a quick price recovery can also be the result of a small resilience and a favorable fundamental price move. We find that the optimal trade execution strategy is *aggressive in the money*: If prices move in a favorable direction, i.e., price recovery is quick, then the believed likelihood of a high resilience is increased. Therefore trade execution is accelerated in order to reduce market risk. If however prices move in an unfavorable direction, then trading is slowed down, since cost estimates for quick liquidation are corrected upwards. This behavior is in line with traders' intuition: If market prices move unfavorably during a trade execution, then this may be caused by the trader pushing too hard for quick execution and the trader should thus slow down his trading.

Mathematical results

In Chapter 1 to 3 we discuss our limit order book model with time-varying depth. We start by setting up our risk-neutral investor optimal execution framework, which we consider both in discrete and continuous trading time. It turns out that the following issues make this optimal control problem mathematically challenging: First of all, we need to account for the constraint that the order has to be finished at the final time. Second, the resilience effect makes the price impact not only dependent on the current, but also on earlier trades of the large investor. Just as in Obizhaeva and Wang (2006), basic model assumptions guarantee that this dependence can be aggregated into a process that keeps track of the remaining price impact due to earlier trades. The main difference and difficulty compared to Obizhaeva and Wang (2006) is the fact that we go beyond deterministic controls. We have to include adapted trading strategies in order to account for stochastic liquidity.

As in Obizhaeva and Wang (2006), it is necessary to allow not only absolutely continuous processes, but actually to use general càglàd processes of bounded variation to describe our trading trajectories. This is due to the fact that our model turns out to lead to a *singular control problem*. The corresponding nonlinear Hamilton-Jacobi-Bellman (HJB) equation is not accessible by standard literature on partial differential equations.

Therefore, special methods have to be used in the analysis of this problem. Indeed, as in the examples documented, e.g., in Benes, Shepp, and Witsenhausen (1980) and Davis and Norman (1990), we can state closed form solutions only under specific assumptions.

Our considered value function has a four-dimensional domain. It depends on time and three additional state variables: The price impact due to earlier trades, the shares left to be traded and the current depth level of the order book. Corresponding to intuition, one can show that the value function, representing the minimal purchasing costs, is decreasing in the depth level and resilience speed as well as increasing in the other three mentioned dimensions. Due to the time-varying nature of the liquidity, the cost function underlying the value function is not necessarily convex in the strategy. This makes the analysis rather delicate.

In Chapter 1, it turns out that our model assumptions enable us to simplify the task of minimizing the expected costs in several perspectives. Under some mild assumptions on the best bid and ask price, we can show that it is never optimal to trade in the opposite direction, i.e., when the overall goal is to purchase shares, then there is no selling in between. Therefore we can restrict our model to only one side of the order book. But this simplification comes at the cost that we additionally need to constrain our optimal strategy set to monotone trajectories. Having a risk-neutral investor and assuming the fundamental asset price to be a martingale, we can without loss of generality set the fundamental price to zero. The same is true for the permanent price impact, since we assume it to be a constant coefficient being multiplied by the number of shares. Due to arbitrage arguments, this is a reasonable and common assumption as, e.g., explained in Huberman and Stanzl (2005). Another important simplification of the problem results from the assumption that the limit order book is block-shaped, i.e., there is the same number of shares available at each price tick. As a result, we can show that the domain of the value function can indeed be reduced to three dimensions by condensing the first two state variables. More precisely, we do not have to consider the number of shares left to be traded and the price impact due to earlier trades separately, but only the ratio of these two values is important.

Although we do not use it in our following arguments, we formally write down the HJB equation as a guidance. As usual for a singular control problem, it is a variational inequality. We can identify a trading and a no-trading region. In Section 2.1, we define these two regions via the value function and formulate a conjecture concerning the structure of these regions: Intuitively, it should be clear that for fixed time and corresponding order book depth, one should be situated in the no-trading region for low and in the trading region for high values of the number of shares over price impact ratio. There should be a time and depth dependent positive barrier between these two regions. It represents the free boundary corresponding to our singular control problem. Astonishingly, it turns out that this conjecture is not always true as the examples collected in Section 2.5 show. Therefore, it is one of our main intentions to identify the situations in which the conjecture holds. We check it separately for different dynamics of the order book depth.

Section 2.2 starts by recapitulating the *constant depth* discrete and continuous time optimal strategies given in Obizhaeva and Wang (2006) and Alfonsi, Fruth, and Schied (2007), reinterprets them in our framework of trading and no-trading regions and slightly enriches them by additionally stating the case of nonzero initial impact due to earlier trades. Section 2.2 goes ahead by discussing more general *deterministic dynamics* of the order book depth. Exploiting the fact that the value function is piecewise quadratic, we first prove by backward induction that the barrier conjecture holds in discrete time. A barrier between the no-trade and the trade region exists. It uniquely determines the optimal strategy and its numerical computation is straightforward. For continuous trading time and continuous depth dynamic, we get the existence of an optimal strategy by Helly's Theorem. Showing convexity of some functions involved in the discrete time induction, we get uniform convergence as the distance between the discrete trading instances decreases. This way, we can transfer the barrier result from discrete to continuous time. In some situations, one can then use the Euler-Lagrange formalism in order to find a closed form solution of the continuous time barrier function. To the best of our knowledge, it is not clear how to incorporate the constraint of monotone trajectories into the formalism. We therefore only consider a modified problem with the larger optimization set of not necessarily monotone trajectories. The solutions of the original and the modified optimization must coincide, when the modified problem yields a monotone optimal trajectory. We identify these situations in terms of the applied deterministic depth profile.

Section 2.3 deals with the case of the order book depth being a *geometric Brownian motion*. This choice seems natural, since it is one of the simplest positive diffusions. In discrete time we can again use backward induction in order to prove that our conjecture holds in a lot of situations. But as opposed to the deterministic case, it turns out that the conjecture can be violated in cases of time-dependent resilience speed or drift of the geometric Brownian motion. Moreover, we introduce the already mentioned notion of aggressive and passive in the liquidity trades and it turns out that the geometric Brownian motion produces passive in the liquidity behavior. Therefore, it does not seem to be a particularly well suited model for the depth.

This motivates Section 2.4. We consider more *general diffusion dynamics* with time and state-dependent drift and volatility as our model input. We find sufficient conditions in terms of the resilience as well as these drift and volatility parameters such that our cost function is convex in the strategy. This guarantees unique optimal strategies and the existence follows with a Komlós argument. Notably, this convexity also suffices to then prove the barrier conjecture from above in discrete and continuous time. This is one of our main results. We also analyze the mentioned sufficient conditions. For the deterministic and geometric Brownian motion case, these conditions from Section 2.4 are clearly more restrictive than the sufficient conditions for our conjecture from Section 2.2 and 2.3 respectively. For the mean-reverting, positive Cox-Ingersoll-Ross process, however, we find that we can apply the results from Section 2.4 when the mean-reversion speed is small compared to the resilience and the volatility in turn being small compared to the mean-reversion speed.

In Section 2.5 we utilize the insights from Section 2.2 to 2.4 in order to construct situations where the barrier conjecture does not hold. One can prove that the conjecture always holds when trading is only allowed at two trading instances. We know from Section 2.2 that it holds for deterministic depth dynamics. But for a specific binomial model with two states of the world for the depth and three trading instances, we can show that the conjecture does not hold. This counterexample can also be extended to continuous time. For the time-inhomogeneous geometric Brownian motion and the Cox-Ingersoll-Ross process, we find such counterexamples at least in discrete time.

We learn from Chapter 2 that it is hard to find structural results for the trade and no-trade region. Only in very specific cases we can state their shape explicitly. This motivates to discuss numerical issues of our singular control problem in Chapter 3. Our aim is to approximate the value function and the free boundary. To the best of our knowledge, there is no systematic approach that treats numerical schemes for HJB variational inequalities in several dimensions directly. Therefore, we use the well-established *Markov chain approximation method* introduced by Kushner and his co-authors as an alternative to a verification argument. An introduction to the method is given in Kushner and Dupuis (2001). Instead of approximating the HJB equation, the control problem itself is approximated. In other words, the value function is approximated on a grid and the state space dynamics are replaced by transition probabilities between the grid points. A finite difference scheme of the HJB equation is used to choose these transition probabilities consistently with the original state dynamics.

Closely following the lines of Kushner and Martins (1991) and the related work of Budhiraja and Ross (2007), we show that the approximated value function converges to the original one as the grid size decreases to zero. The proof is by probabilistic methods and it turns out that the positivity of the transition probabilities is the only assumption for the convergence result. The essential steps in the proof are to truncate the state space and to use tightness results. Due to the control problem being singular, it is necessary to do a time rescaling. Both Kushner and Martins (1991) and Budhiraja and Ross (2007) consider an infinite horizon singular control problem with two-dimensional state space and two fixed vectors to control the state space dynamics. In our case, we need to adjust their convergence proof, since our problem has a finite time horizon, a three-dimensional state space and most notably the control direction is state space dependent. Moreover, our cost structure is qualitatively different, which e.g. complicates the state space truncation. Although the HJB equation is not used in the proof at all, it turns out in Section 3.2.2 that the numerical scheme resulting from the Markov chain method is in some sense equivalent to the implementation of the HJB equation by a finite difference scheme.

Chapter 4 contains the stochastic resilience case. We focus on economic and qualitative findings. From a mathematical point of view, we stick to a model with three trading instances such that our optimization can be done by direct calculations. The filtering aspects turn out to be rather involved and make an insightful analysis of more complex models seem beyond reach with today's methods.

Acknowledgements

First I would like to thank my advisor Peter Bank for his support and ideas over the last years. Moreover, I am particularly indebted to Mikhail Urusov and Torsten Schöneborn. Torsten suggested the stochastic liquidity project originally and for instance suggested the induction argument for the barrier conjecture in the deterministic, discrete time case and the related basic example where the barrier conjecture does not hold. Mikhail took a lot of time to share his mathematical knowledge with me. We worked together on the closed form solutions in the deterministic case and on the existence and uniqueness result in the general setting.

I would also like to thank Ulrich Horst and Steven Shreve for readily accepting the task of being co-examiners. I greatly appreciate their comments.

I am particularly grateful to my colleagues at Deutsche Bank Quantitative Products Laboratory, especially Marcel Höschler, Felix Naujokat and Almut Birsner. They have always been a great source of support and motivation. As one of the first employees at the Quantitative Products Laboratory, I was able to benefit from its outstanding working atmosphere, infrastructure and funding over several years. Therefore, I would like to sincerely thank the people from Humboldt Universität, Technische Universität Berlin and Deutsche Bank, in particular Alexander Schied and Marcus Overhaus, for making this joint project possible. Moreover, the support by Berlin Mathematical School and Technische Universität Berlin is acknowledged.

Last but not least, I wish to express my deepest gratitude to my family, friends and foremost my husband Tommy. As so often, their encouragement has been invaluable.

Chapter 1

Model description and preparations

In practice, very large orders are often split into a number of consecutive orders to reduce the overall price impact. Hence, the question at hand is to determine the size of the individual orders so as to minimize a cost criterion. We call this the optimal execution problem of a large investor and consider the specific case where this investor is risk-neutral. In order to discuss this optimization problem, we are going to specify a model both for the unaffected price and the price impact. The price impact model, that we are going to use, allows for the price impact to be stochastic. This is our main contribution. Our model is based on the work of Obizhaeva and Wang (2006). The main idea is to derive the impact model from a limit order book.

1.1 Order book dynamics and assumptions

The considered one-asset model derives its price dynamics from a limit order book that is exposed to repeated market orders of a large investor (sometimes referred to as the trader). A limit order book is a collection of the limit orders of all market participants in an electronic market. Each limit order has the number of shares, that the market participant wants to trade, and a price per share attached to it. The price represents a minimal price in case of a sell and a maximal price in case of a buy order. Compared to a limit order, a market order does not have an attached price per share, but instead is executed immediately against the best limit orders waiting in the book. Thus, there is a tradeoff between price saving and immediacy when using limit and market orders.

The goal of the investor is to use market orders in order to purchase a large amount x of shares within a certain time period $[0, T]$, where T typically ranges from a few hours up to a few trading days. On this macroscopic time scale, the restriction to market orders is not severe. A subsequent consideration of small time windows including limit order trading is common practice in banks. See Naujokat and Westray (2011) for a discussion of a large investor execution problem where both market and limit orders

are allowed. In our case, emphasis is on buy orders, and we first concentrate on the upper part of the limit order book. It consists of limit sell orders offered at various ask prices. The lowest ask price is called the *best ask price*.

Suppose first that the trader is not active. We assume that the corresponding unaffected best ask price A^u is a càdlàg martingale on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$. Moreover, $A_0^u = A_0$ for a constant A_0 and A^u is an \mathcal{H}^1 -martingale, i.e. $\mathbb{E}\sqrt{[A^u, A^u]_T} < \infty$, or, equivalently, $\mathbb{E}[\sup_{t \in [0, T]} |A_t^u|] < \infty$. This assumption includes in particular the case of the Bachelier model, i.e., $A_t^u = A_0^u + \sigma W_t^A$ for an (\mathcal{F}_t) -Brownian motion W^A , as considered in Obizhaeva and Wang (2006). We emphasize, however, that we can take any \mathcal{H}^1 -martingale and hence use, e.g., a driftless geometric Brownian motion, which avoids the counterintuitive negative prices of the Bachelier model. Moreover, we can allow for jumps in the dynamics of A^u so as to model the trading activities of other large investors in the market.

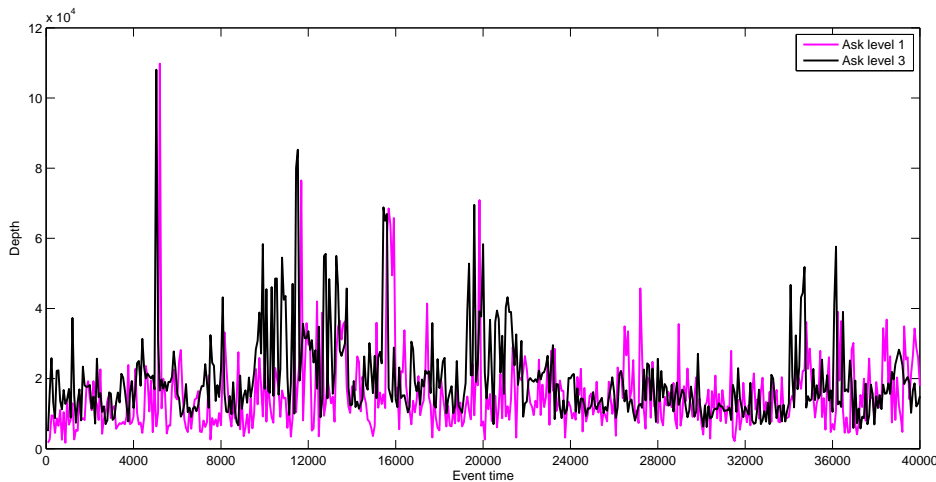


Figure 1.1: Time series of market depth in the order book of Fortis, Euronext, Amsterdam, August 1, 2008. Taken from Hautsch and Huang (2010).

Let us consider real-life order book dynamics in the price per share and in time. Figure 1.1 taken from Hautsch and Huang (2010) shows a typical dynamic. The depth, represented by the number of shares on the first and third ask level or ask tick respectively, is quite volatile during one trading day. Moreover, the depth is mean-reverting and there is evidence for co-movement between the individual tick levels. Although the order book height behind the market is typically greater than that directly at the spread, we follow Obizhaeva and Wang (2006) in assuming a constant ask price distribution in our model: The number of shares offered at prices in the interval $[A_t^u, A_t^u + \Delta A]$ is given by $q_t \cdot \Delta A$ for the order book height $q_t > 0$. Here, we are interested in the dynamics of the order book in time. That is why we assume the book to be block-shaped meaning that the same number of shares q_t is available at each price level. Cont, Kukanov, and Stoikov (2010) empirically find that such a block shape performs well. See Alfonsi, Fruth, and Schied (2010) and Predoiu, Shaikhet, and Shreve (2011) for non-constant shapes of the book in the price per share. They conclude that the

optimal execution strategy of the investor is robust with respect to the order book shape. In Obizhaeva and Wang (2006), Alfonsi, Fruth, and Schied (2010) and Predoiu, Shaikhet, and Shreve (2011) the shape is assumed to stay constant over time. That is in Obizhaeva and Wang (2006) the order book height is chosen as a constant $q_t \equiv q$ while we assume an exogenously given stochastic dynamic for q_t .

Suppose for instance that our trader places a buy market order of $\xi_0 > 0$ shares at time $t = 0$. This market order consumes all shares located at prices between A_0 and $A_0 + D_{0+}$, where D_{0+} is determined by

$$D_{0+} \cdot q_0 = \xi_0.$$

Consequently, the ask price is shifted up from A_0 to

$$A_{0+} := A_0 + D_{0+}.$$

Let us denote by A_t the actual best ask price at time t , i.e., the ask price after taking the price impact of previous buy market orders of the trader into account. Define

$$D_t := A_t - A_t^u$$

as the price impact or *extra spread* caused by the past actions of the trader. Another buy market order of $\xi_t > 0$ shares now consumes all the shares offered at prices between A_t and

$$A_{t+} := A_t + D_{t+} - D_t = A_t^u + D_{t+},$$

where D_{t+} is determined by the condition

$$(D_{t+} - D_t) \cdot q_t = \xi_t.$$

Thus, the process D captures the impact of market orders on the current best ask price, see Figure 1.2.

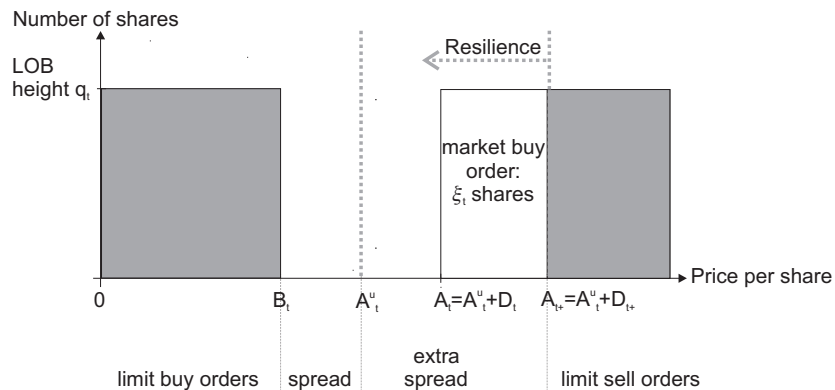


Figure 1.2: Snapshot of the block-shaped order book model at time t .

We still need to specify how D evolves when our trader is inactive between market orders. It is a well established empirical fact, that order books exhibit a certain resilience

as to the price impact of large block orders. That is, only a fraction of the immediate impact ξ_t/q_t is permanent, while the remaining fraction is temporary and decays to zero. One can distinguish spread and depth resilience, see e.g. Kempf, Mayston, and Yadav (2007). In modeling this resilience, we assume spread resilience to be prevailing and consider an exponential recovery of the temporary part of the impact with a fixed time-dependent, deterministic recovery rate ρ_t , which we assume to be a strictly positive integrable function on $[0, T]$. Such an extension of their original model with constant recovery rate was suggested by Obizhaeva and Wang (2006, Section 8.1). Weiss (2010) also considers exponential resilience and shows that the results of Alfonsi, Fruth, and Schied (2010) and in particular Obizhaeva and Wang (2006) can be adapted when the recovery rate depends on the trade size of the large investor. Alfonsi, Schied, and Slynko (2009) recently considered more general deterministic decay functions than the exponential one in a model with constant order book height. For a discussion of a stochastic recovery rate ρ , we refer to Chapter 4.

In our model, the price impact at time $s \geq t$ of a buy market order $\xi_t > 0$ placed at time t is assumed to be

$$\gamma\xi_t + K_t e^{-\int_t^s \rho_u du} \xi_t,$$

where the positive constant γ quantifies the magnitude of the permanent impact and K_t is the temporary impact coefficient. Notice that this temporary impact model is different from the one which is used, e.g., in Almgren and Chriss (2001) and Almgren (2003). It slowly decays to zero instead of vanishing immediately and thus prices depend on previous trades. For the order book height we have

$$q_t = (\gamma + K_t)^{-1}.$$

Throughout this thesis, we discuss results for various assumptions on K . All types of assumptions on this illiquidity process K are collected here in order to refer to them in the following chapters.

- *Assumption Basic:* $(K_t)_{t \in [0, T]}$ is a strictly positive, adapted, integrable process with first moments.
- *Assumption Const:* K is constant to $\kappa > 0$.
- *Assumption Determ:* $K : [0, T] \rightarrow (0, \infty)$ is deterministic and Lebesgue integrable.
- *Assumption CtsDeterm:* $K : [0, T] \rightarrow (0, \infty)$ is deterministic and continuous.
- *Assumption Smooth:* $K : [0, T] \rightarrow (0, \infty)$ is deterministic, two times continuously differentiable and ρ is continuously differentiable.

- *Assumption GBM*: K is a geometric Brownian motion, $K_0 > 0$,

$$dK_t = \bar{\mu}(t)K_t dt + \bar{\sigma}(t)K_t dW_t^K$$

for an (\mathcal{F}_t) -Brownian motion W^K and $\bar{\mu}, \bar{\sigma}^2$ deterministic, integrable functions.

- *Assumption SpecialGBM*: K is a geometric Brownian motion, $K_0 > 0$,

$$dK_t = \bar{\mu}(t)K_t dt + \bar{\sigma}(t)K_t dW_t^K$$

for an (\mathcal{F}_t) -Brownian motion W^K and $\bar{\mu}, \bar{\sigma}$ deterministic, integrable functions.

For fixed trading instances $0 = t_0 < \dots < t_N = T$, either $\int_{t_n}^T (\bar{\mu}_s + \rho_s) ds \leq 0$ for all $n = 0, \dots, N-1$ or $\int_{t_n}^{t_{n+1}} (\bar{\mu}_s + \rho_s) ds > 0$ for all $n = 0, \dots, N-1$.

- *Assumption Diff*: K is a time-inhomogeneous diffusion, $K_0 > 0$,

$$dK_t = \mu(t, K_t)dt + \sigma(t, K_t)dW_t^K$$

for an (\mathcal{F}_t) -Brownian motion W^K and $\mu, \sigma : [0, T] \times (0, \infty) \rightarrow \mathbb{R}$ such that the stochastic differential equation has a strong solution which is unique in law, has first moments and is strictly positive, integrable a.s.

- *Assumption SpecialDiff*: K is a time-inhomogeneous diffusion, $K_0 > 0$,

$$dK_t = \mu(t, K_t)dt + \sigma(t, K_t)dW_t^K$$

for an (\mathcal{F}_t) -Brownian motion W^K and $\mu, \sigma : [0, T] \times (0, \infty) \rightarrow \mathbb{R}$ such that the stochastic differential equation has a strong solution which is unique in law, has first moments and is strictly positive, integrable a.s. Furthermore,

$$\text{i) } \eta_t := \frac{2\rho_t}{K_t} + \frac{\mu(t, K_t)}{K_t^2} - \frac{\sigma^2(t, K_t)}{K_t^3} > 0 \quad \text{for all } t \in [0, T],$$

$$\text{ii) } \mathbb{E} \left[\frac{\sup_{t \in [0, T]} K_t^2}{\inf_{t \in [0, T]} K_t} \right] < \infty,$$

$$\text{iii) } \mathbb{E} \left[\left(\int_0^T |\eta_t| dt \right) \left(\sup_{t \in [0, T]} K_t^2 \right) \right] < \infty.$$

- *Assumption HomogDiff*: K is a time-homogeneous diffusion, $K_0 \geq 0$,

$$dK_t = \mu(K_t)dt + \sigma(K_t)dW_t^K$$

for an (\mathcal{F}_t) -Brownian motion W^K , continuous μ and locally bounded $\sigma : [0, \infty) \rightarrow \mathbb{R}$ such that the stochastic differential equation has a strong solution which is unique in law, positive, integrable a.s. and $\mathbb{E} [\sup_{t \in [0, T]} K_t] < \infty$.

- *Assumption SpecialCIR*: K is a Cox-Ingersoll-Ross process, $K_0 > 0$,

$$dK_t = \bar{\mu}(\bar{K} - K_t)dt + \bar{\sigma}\sqrt{K_t}dW_t^K$$

for an (\mathcal{F}_t) -Brownian motion W^K . Further assume strictly positive, constant resilience speed ρ ,

$$2\rho \geq \bar{\mu} > 0 \quad \text{and} \quad \bar{\mu}\bar{K} > 2\bar{\sigma}^2.$$

Assumption Basic is the basic assumption that follows from all other given assumptions on K . It states the minimal requirements for our problem to be economically sensible and well-defined. In Section 2.2, we consider the special case of deterministic K . First of all, we summarize the case of constant K . We then work under *Assumption Determ* and consider discrete trading time. For the deterministic continuous time case, we need *Assumption CtsDeterm* for the existence of optimal strategies. Having continuously differentiable processes K , we can in some cases explicitly compute these optimal strategies.

In Section 2.3, we work under *Assumption SpecialGBM* in order to characterize optimal strategies in discrete time for K being a geometric Brownian motion. Section 2.4 deals with both discrete and continuous time trading and quite general time-inhomogeneous diffusions for K . In this context, *Assumption SpecialDiff i), ii), iii)* is used in order to prove existence and uniqueness of optimal strategies. *Assumption HomogDiff* is applied in the numerical treatment of our optimization problem in Section 3.1. The example of a Cox-Ingersoll-Ross process in *Assumption SpecialCIR* both satisfies *Assumption SpecialDiff* and *HomogDiff* and is used for some illustrations in Section 3.3.

Let us continue with the model specification. The process D can be computed as the cumulative price impact of all past buy market orders. That is, if buy market orders ξ_{t_n} are placed at times t_n , then

$$D_t = \gamma \sum_{t_n < t} \xi_{t_n} + \sum_{t_n < t} K_{t_n} e^{-\int_{t_n}^t \rho_s ds} \xi_{t_n}.$$

We want to formulate our model for general trading strategies in continuous time and get

$$dD_t = -\rho_t(D_t - \gamma\Theta_t)dt + (\gamma + K_t)d\Theta_t$$

with the adapted process (Θ_t) of finite variation describing the number of shares that the trader holds at time t . Notice that the process D depends on Θ , although this is not explicitly marked in its notation.

Let us now go ahead by describing the cost minimization problem of the trader. When placing a single buy market order of size $\xi_t \geq 0$ at time t , he purchases at prices $A_t^u + d$, with d ranging from D_t to D_{t+} , see Figure 1.2. Due to the block-shaped limit order book, the total costs of the buy market order amount to

$$(A_t^u + D_t) \xi_t + \frac{D_{t+} - D_t}{2} \xi_t = (A_t^u + D_t) \xi_t + \frac{\xi_t^2}{2q_t}.$$

In other words, the total costs are the number of shares times the average price per share $(A_t^u + D_t + \frac{\xi_t}{2q_t})$.

Let

$$\mathcal{A}_t := \left\{ \begin{array}{l} \Theta : \Omega \times [t, T+] \rightarrow [0, \infty) \\ \mathcal{F}_s - \text{adapted, increasing, bounded, càglàd with } \Theta_t = 0 \text{ a.s.} \end{array} \right\}$$

be the set of *increasing* strategies Θ with $\Delta\Theta_s := \Theta_{s+} - \Theta_s$. In particular, trading in rates and impulse trades are allowed. Notice that a strategy from \mathcal{A}_t consists of a left-continuous process $(\Theta_s)_{s \in [t, T]}$ and an additional random variable Θ_{T+} with $\Delta\Theta_T = \Theta_{T+} - \Theta_T \geq 0$ being the last trade of the strategy. Denote by

$$\mathcal{A}_t(x) := \{\Theta \in \mathcal{A}_t | \Theta_{T+} = x \text{ a.s.}\} \quad (1.1)$$

the *admissible strategies* that build up a position of $x \in [0, \infty)$ shares by time T almost surely.

Up until now, we have only described the effect of buy orders on the ask side of the order book. When a client of a financial institution wants to purchase $x > 0$ shares up to time T , a restriction to buy orders is not only plausible, but is also a necessity due to legal issues or a requirement of the client. In the following, we also give a mathematical argument showing that under mild assumptions it is not beneficial to sell.

Assume that we keep track of $\Theta, \tilde{\Theta} \in \mathcal{A}_0$, where Θ is the number of shares traded by market buy orders and $\tilde{\Theta}$ represents the shares that the trader has sold using market sell orders. Analogously to buy orders, sell orders are executed at the best bid price B_t minus the ad-hoc impact $\frac{\Delta\tilde{\Theta}_t}{2q_t}$. We allow the best bid to depend on the unaffected best ask and the strategy $(\Theta, \tilde{\Theta})$ of the large investor as long as it satisfies (1.5).

Our aim is to minimize the *total expected costs*

$$\inf_{\Theta, \tilde{\Theta} \in \mathcal{A}_0, \Theta_{T+} - \tilde{\Theta}_{T+} = x} \mathbb{E} \left[\int_{[0, T]} \left(A_t + \frac{\Delta\Theta_t}{2q_t} \right) d\Theta_t - \int_{[0, T]} \left(B_t - \frac{\Delta\tilde{\Theta}_t}{2q_t} \right) d\tilde{\Theta}_t \right] \quad (1.2)$$

with

$$A_t = A_t^u + D_t$$

and the martingale A^u and the ask side impact D being described above. That is we imagine the considered large investor to be risk-neutral and we allow his optimal strategy to consist of both sell and buy orders. But it turns out that under condition (1.5) on the best bid, it is never optimal to do intermediate sell orders when the overall goal is the purchase of $x > 0$ shares. Before proving the corresponding Proposition 1.1.2, we need Proposition 1.1.1 as a preparation. It deals with the optimization problem

$$\inf_{\Theta \in \mathcal{A}_0(x)} \mathbb{E} \left[\int_{[0, T]} \left(A_t + \frac{\Delta\Theta_t}{2q_t} \right) d\Theta_t \right] \quad (1.3)$$

that we get when we restrict (1.2) to buy strategies. Proposition 1.1.1 verifies that the expected costs in (1.3) resulting from the unaffected best ask price and permanent impact do not depend on the strategy. Therefore, we can, without loss of generality, set $A^u \equiv 0$, $\gamma = 0$. Consequently, only the expected temporary impact needs to be minimized over all admissible buying strategies.

Proposition 1.1.1. (Only temporary impact has to be considered).

For all $\Theta \in \mathcal{A}_0(x)$,

$$\mathbb{E} \left[\int_{[0,T]} \left(A_t + \frac{\Delta\Theta_t}{2q_t} \right) d\Theta_t \right] = A_0x + \frac{\gamma}{2}x^2 + \mathbb{E} \left[\int_{[0,T]} \left(D_t^{\gamma=0} + \frac{K_t}{2}\Delta\Theta_t \right) d\Theta_t \right]$$

with $D_t^{\gamma=0} := D_t - \gamma\Theta_t$ solving $dD_t^{\gamma=0} = -\rho_t D_t^{\gamma=0} dt + K_t d\Theta_t$.

Proof. We start by looking at the expected costs caused by the unaffected best ask price martingale. Using integration by parts for càglàd processes as well as the facts that $\Theta \in \mathcal{A}_0(x)$ is bounded and that A^u is an \mathcal{H}^1 -martingale yields

$$\mathbb{E} \left[\int_{[0,T]} A_t^u d\Theta_t \right] = \mathbb{E} [A_{T+}^u \Theta_{T+} - A_0^u \Theta_0] = A_0x. \quad (1.4)$$

Let us now turn to the simplification of our optimization problem due to permanent impact. To this end, we differentiate between the temporary price impact $D_t^{\gamma=0}$ and the total price impact $D_t = D_t^{\gamma=0} + \gamma\Theta_t$ that we get by adding the permanent impact. We can then write

$$\begin{aligned} & \int_{[0,T]} \left(D_t + \frac{\Delta\Theta_t}{2q_t} \right) d\Theta_t \\ &= \int_{[0,T]} \left(\left[- \int_{[0,t)} \rho_s D_s^{\gamma=0} ds + \int_{[0,t)} (\gamma + K_s) d\Theta_s \right] + \frac{(\gamma + K_t)\Delta\Theta_t}{2} \right) d\Theta_t \\ &= \frac{\gamma}{2}x^2 + \int_{[0,T]} \left(D_t^{\gamma=0} + \frac{K_t}{2}\Delta\Theta_t \right) d\Theta_t, \end{aligned}$$

since integration by parts for càglàd processes and $\Theta \in \mathcal{A}_0(x)$ yield

$$\int_{[0,T]} \left(\Theta_t + \frac{\Delta\Theta_t}{2} \right) d\Theta_t = \frac{\Theta_{T+}^2 - \Theta_0^2}{2} = \frac{x^2}{2}.$$

□

Consider the expected costs of a round trip $\eta \in \mathbb{R}_{>0}$. That is we compare the revenue of a sale η at time t with the cost of buying back η shares later at time $s \geq t$. Due to $D_s \geq \gamma\Theta_t$ and the martingale assumption we get

$$\mathbb{E} \left[- \left(B_t - \frac{\eta}{2q_t} \right) \eta + \left(A_s^u + D_s + \frac{\eta}{2q_s} \right) \eta \right] \geq \mathbb{E} [(A_t^u + \gamma\Theta_t - B_t) \eta].$$

These expected costs of a round trip should be positive. This motivates to assume that for all $t \in [0, T]$ and $\Theta, \tilde{\Theta} \in \mathcal{A}_0$ the best bid price satisfies

$$B_t = B_t \left(A_t^u, \Theta_{[0,t]}, \tilde{\Theta}_{[0,t]} \right) \leq A_t^u + \gamma(\Theta_t - \tilde{\Theta}_t). \quad (1.5)$$

Taking this reasonable bid price assumption, we can prove in Proposition 1.1.2 that “not selling” is an optimal sell strategy in the original optimization problem (1.2). We do so by showing that (1.2) is greater than or equal to (1.3) meaning that the two terms must coincide.

Before we do so, let us economically justify (1.5). For a moment, ignore the temporary impact. We can do this, since the temporary impact would even increase the difference between the bid and the ask price. If we had to choose one fair price, it would be canonical to take the martingale plus the net permanent impact, i.e. $A_t^u + \gamma(\Theta_t - \tilde{\Theta}_t)$. Actually, we have two prices instead of one and it seems natural to assume that the ask price should be larger than $A_t^u + \gamma(\Theta_t - \tilde{\Theta}_t)$, which is satisfied in our model, and the bid price should be smaller as given in (1.5).

Proposition 1.1.2. (No trades in the opposite direction).

Under the bid price assumption (1.5), the terms (1.2) and (1.3) coincide.

Proof. Consider $\Theta, \tilde{\Theta} \in \mathcal{A}_0$ with $\Theta_{T+} - \tilde{\Theta}_{T+} = x$. Without loss of generality we can assume $\Delta\Theta_t \Delta\tilde{\Theta}_t = 0$ for all $t \in [0, T]$. Making use of (1.5) yields

$$\begin{aligned}
& \mathbb{E} \left[\int_{[0,T]} \left(A_t + \frac{\Delta\Theta_t}{2q_t} \right) d\Theta_t \right] - \mathbb{E} \left[\int_{[0,T]} \left(B_t \left(A_t^u, \Theta_{[0,t]}, \tilde{\Theta}_{[0,t]} \right) - \frac{\Delta\tilde{\Theta}_t}{2q_t} \right) d\tilde{\Theta}_t \right] \\
& \geq \mathbb{E} \left[\int_{[0,T]} \left(A_t^u + \gamma\Theta_t + D_t^{\gamma=0} + \frac{\gamma}{2}\Delta\Theta_t + \frac{K_t}{2}\Delta\Theta_t \right) d\Theta_t \right] \\
& \quad - \mathbb{E} \left[\int_{[0,T]} \left(A_t^u + \gamma\Theta_t - \gamma\tilde{\Theta}_t - \frac{\gamma}{2}\Delta\tilde{\Theta}_t - \frac{K_t}{2}\Delta\tilde{\Theta}_t \right) d\tilde{\Theta}_t \right] \\
& \geq \mathbb{E} \left[\int_{[0,T]} A_t^u d(\Theta_t - \tilde{\Theta}_t) \right] \\
& \quad + \gamma \mathbb{E} \left[\int_{[0,T]} \left(\Theta_t + \frac{\Delta\Theta_t}{2} \right) d\Theta_t + \int_{[0,T]} \left(\tilde{\Theta}_t - \Theta_t + \frac{\Delta\tilde{\Theta}_t}{2} \right) d\tilde{\Theta}_t \right] \\
& \quad + \mathbb{E} \left[\int_{[0,T]} \left(D_t^{\gamma=0} + \frac{K_t}{2}\Delta\Theta_t \right) d\Theta_t \right].
\end{aligned}$$

Analogously to (1.4), the first of these terms equals A_0x since $\Theta, \tilde{\Theta}$ are monotonic and bounded. For the second one we do integration by parts to deduce

$$\begin{aligned}
& \int_{[0,T]} (2\Theta_t + \Delta\Theta_t) d\Theta_t + \int_{[0,T]} (2\tilde{\Theta}_t + \Delta\tilde{\Theta}_t) d\tilde{\Theta}_t - 2 \int_{[0,T]} \Theta_t d\tilde{\Theta}_t \\
& = \Theta_{T+}^2 + \tilde{\Theta}_{T+}^2 - 2\Theta_{T+}\tilde{\Theta}_{T+} + 2 \int_{[0,T]} \tilde{\Theta}_t d\Theta_t \geq (\Theta_{T+} - \tilde{\Theta}_{T+})^2 = x^2.
\end{aligned}$$

The assertion follows thanks to Proposition 1.1.1. \square

We want to point out that our approach with condition (1.5) is different from Alfonsi, Schied, and Slynko (2009), Gatheral, Schied, and Slynko (2009) and Gatheral, Schied,

and Slynko (2010) where it is assumed that there is no spread. Buying and selling occurs at the same price. The output of the optimal execution problem is then used as a sanity check for the input, i.e., the applied price impact model. In case of the optimal strategy being non-monotonic, the impact model is rejected.

Our final optimization problem accounting for both the simplifications given in Proposition 1.1.1 and 1.1.2 is summarized in the next section.

1.2 Summary of the singular control problem

Let us now summarize the control problem that we consider in the following. Due to Proposition 1.1.1 and 1.1.2 we can without loss of generality set $A^u \equiv 0$, $\gamma = 0$, $\tilde{\Theta} \equiv 0$ and define the *cost function* $J : [0, T] \times [0, \infty) \times \mathcal{A}_0 \times (0, \infty) \rightarrow [0, \infty)$ as

$$J(\Theta) := J(t, \delta, \Theta, \kappa) := \mathbb{E}_{t, \delta, \kappa} \left[\int_{[t, T]} \left(D_s + \frac{K_s}{2} \Delta \Theta_s \right) d\Theta_s \right]. \quad (1.6)$$

We denote by Θ_s the number of shares hold by the large investor at time $s \in [t, T]$ and the function J represents the total expected temporary impact costs on the time interval $[t, T]$ when $D_t = \delta$, $K_t = \kappa$. The process D stands for the price impact, i.e., the deviation of the current ask price from its steady state level and K is the price impact coefficient satisfying *Assumption Basic*. The deviation D_s results from past trades on $[0, s]$ in the following way

$$dD_s = -\rho_s D_s ds + K_s d\Theta_s, \quad D_t = \delta. \quad (1.7)$$

That is for $s \in [t, T]$

$$D_s = \int_t^s K_u e^{-\int_u^s \rho_r dr} d\Theta_u + \delta e^{-\int_t^s \rho_u du}. \quad (1.8)$$

The resilience speed $\rho : [t, T] \rightarrow (0, \infty)$ is assumed to be deterministic, strictly positive and Lebesgue integrable. The process K describes the externally given dynamics of the depth and D represents the movement of the order book block due to the trades of the large investor and the resilience effect. Notice that although K is not influenced by the investor's strategy, the liquidity in the market does react to the strategy via the resilience.

The trader can choose actions from the set of *admissible strategies* $\mathcal{A}_t(x)$ in (1.1) that build up a position of $x \in [0, \infty)$ shares by time T almost surely. Let us now define our *value function for continuous trading time* $U : [0, T] \times [0, \infty)^2 \times (0, \infty) \rightarrow [0, \infty)$ as

$$U(t, \delta, x, \kappa) := \inf_{\Theta \in \mathcal{A}_t(x)} J(t, \delta, \Theta, \kappa). \quad (1.9)$$

Remark 1.2.1. (Optimization over deterministic instead of adaptive strategy set). Imagine that we would constrain our optimization set to deterministic strategies only. That is we would not allow any reaction of our optimal strategy to the dynamic of the order book height. This would correspond to a modified optimization problem (1.9) with $\mathbb{E}_{t,\delta,\kappa}[K_s]$ instead of K_s as our illiquidity process, since D is linear in K . That is we would be in a setting with deterministic K .

Assume that trading is only allowed at given discrete trading times

$$0 = t_0 < t_1 < \dots < t_N = T$$

and define $\tilde{n}(t) := \inf\{n = 0, \dots, N | t_n \geq t\}$. We then have to constrain our admissible strategy set to

$$\begin{aligned} \mathcal{A}_t^N := \{ \Theta \in \mathcal{A}_t \mid & \Theta_s = 0 \text{ on } [t, t_{\tilde{n}(t)}] \text{ and} \\ & \Theta_s = \Theta_{t_{n+}} \text{ a.s. on } (t_n, t_{n+1}) \text{ for } n = \tilde{n}(t), \dots, N-1 \} \subset \mathcal{A}_t \end{aligned}$$

and the *value function for discrete trading time* becomes

$$U^N(t, \delta, x, \kappa) := \inf_{\Theta \in \mathcal{A}_t^N(x)} J(t, \delta, \Theta, \kappa) \geq U(t, \delta, x, \kappa) \quad (1.10)$$

with $\mathcal{A}_t^N(x) := \{ \Theta \in \mathcal{A}_t^N | \Theta_{T+} = x \text{ a.s.} \}$. Introducing the discrete trade at time t as $\xi_t := \Delta\Theta_t$ and slightly abusing notation $\xi_n := \xi_{t_n}$, we can also write the cost integral as a sum

$$U^N(t, \delta, x, \kappa) = \inf_{\Theta \in \mathcal{A}_t^N(x)} \mathbb{E}_{t,\delta,\kappa} \left[\sum_{t_n \geq t} \left(D_{t_n} + \frac{K_{t_n}}{2} \xi_n \right) \xi_n \right]. \quad (1.11)$$

Both value functions $u = U$ and $u = U^N$ admit the *boundary conditions*

$$u(T, \delta, x, \kappa) = \left(\delta + \frac{\kappa}{2} x \right) x \text{ and } u(t, \delta, 0, \kappa) = 0. \quad (1.12)$$

Convention: Instead of writing u and meaning that this result holds for both U and U^N , we only write U and still mean that the statement is also true for U^N if not stated otherwise. This procedure applies analogously to \mathcal{A}_t and \mathcal{A}_t^N .

In the following, it turns out to be useful that the discrete time value function satisfies the *dynamic programming principle*, i.e.,

$$\begin{aligned} U^N(t_n, \delta, x, \kappa) = & \quad (1.13) \\ \inf_{\xi \in [0, x]} \left\{ \left(\delta + \frac{\kappa}{2} \xi \right) \xi + \mathbb{E}_{t_n, \delta, \kappa} \left[U^N \left(t_{n+1}, \left(\delta + \kappa \xi \right) e^{-\int_{t_n}^{t_{n+1}} \rho_s ds}, x - \xi, K_{t_{n+1}} \right) \right] \right\} \end{aligned}$$

for $n = 0, \dots, N-1$. See, e.g., Bertsekas and Shreve (1978) for a justification of the dynamic programming principle and especially measurability issues. Let us outline how

one derives (1.13). We consider (1.11), notice that $\left(D_{t_n} + \frac{K_{t_n}}{2}\xi_n\right)\xi_n$ is \mathcal{F}_{t_n} measurable and apply the tower property of the conditional expectation to get

$$\begin{aligned} U^N(t_n, \delta, x, \kappa) &= \\ & \inf_{\Theta \in \mathcal{A}_{t_n}^N(x)} \left\{ \left(\delta + \frac{\kappa}{2}\xi_n \right) \xi_n \right. \\ & \left. + \mathbb{E}_{t_n, \delta, \kappa} \left[\mathbb{E}_{t_{n+1}, D_{t_{n+1}}, K_{t_{n+1}}} \left[\sum_{j=n+1}^N \left(D_{t_j} + \frac{K_{t_j}}{2}\xi_j \right) \xi_j \right] \right] \right\} \\ &= \inf_{\xi_n \in [0, x]} \left\{ \left(\delta + \frac{\kappa}{2}\xi_n \right) \xi_n \right. \\ & \left. + \mathbb{E}_{t_n, \delta, \kappa} \left[\inf_{\Theta \in \mathcal{A}_{t_{n+1}}^N(x-\xi_n)} \mathbb{E}_{t_{n+1}, D_{t_{n+1}}, K_{t_{n+1}}} \left[\sum_{j=n+1}^N \left(D_{t_j} + \frac{K_{t_j}}{2}\xi_j \right) \xi_j \right] \right] \right\}. \end{aligned}$$

The last equation is essential. It holds due to the Markovian structure of our problem, i.e., the conditional expectation given the information up to time t_{n+1} does not depend on the past, but only on the values at time t_{n+1} .

As preparation for following results, we state some easy to check comparative statics satisfied by the value function. The value function is increasing in t, δ, x and the price impact coefficient K as well as decreasing with respect to the resilience speed function.

Lemma 1.2.2. (Comparative statics for the value function).

Assume that an optimal strategy exists.

a) *Then the value function U is increasing in t in the sense that for all $0 \leq t_1 \leq t_2 \leq T$*

$$U(t_1, \delta, x, \kappa) \leq \mathbb{E}_{t_1, \kappa} [U(t_2, \delta, x, K_{t_2})].$$

b) *Then the value function U is strictly increasing in δ, x .*

c) *Fix $t \in [0, T]$. For $\kappa > 0$ denote by $K^{t, \kappa}$ the process on $[t, T]$ that satisfies $K_t^{t, \kappa} = \kappa$. Assume that $0 < K_s^{t, \kappa_1} \leq K_s^{t, \kappa_2}$ a.s. for all $s \in [t, T]$. Then*

$$U(t, \delta, x, \kappa_1) \leq U(t, \delta, x, \kappa_2).$$

d) *Fix $t \in [0, T]$. Assume that two resilience speed functions $\hat{\rho}, \check{\rho} : [t, T] \rightarrow (0, \infty)$ satisfying $\hat{\rho}_s \geq \check{\rho}_s$ for all $s \in [t, T]$ are considered. Then the value function corresponding to $\hat{\rho}$ is less than or equal to the one corresponding to $\check{\rho}$.*

Proof.

- a) Suppose for a contradiction that U fails to be increasing in t , i.e., there exist δ, x, κ , $0 \leq t_1 \leq t_2 \leq T$ such that

$$U(t_1, \delta, x, \kappa) > \mathbb{E}_{t_1, \kappa} [U(t_2, \delta, x, K_{t_2})]. \quad (1.14)$$

Let $\Theta^{t_2} \in \mathcal{A}_{t_2}(x)$ denote an optimal strategy in the sense that

$$U(t_2, \delta, x, K_{t_2}) = J(t_2, \delta, \Theta^{t_2}, K_{t_2}). \quad (1.15)$$

Define $\Theta^{t_1} \in \mathcal{A}_{t_1}(x)$ via

$$\Theta_s^{t_1} := \begin{cases} 0 & \text{for } s \in [t_1, t_2] \\ \Theta_s^{t_2} & \text{for } s \in (t_2, T] \end{cases}.$$

The definition of the value function then yields

$$U(t_1, \delta, x, \kappa) \leq J(t_1, \delta, \Theta^{t_1}, \kappa) = \mathbb{E}_{t_1, \kappa} \left[J \left(t_2, \delta e^{-\int_{t_1}^{t_2} \rho_s ds}, \Theta^{t_2}, K_{t_2} \right) \right].$$

As we see in (1.8), for each $s \in [t, T]$ and fixed strategy Θ , the price impact D_s is increasing in δ such that also J must be increasing in δ . Therefore

$$\mathbb{E}_{t_1, \kappa} \left[J \left(t_2, \delta e^{-\int_{t_1}^{t_2} \rho_s ds}, \Theta^{t_2}, K_{t_2} \right) \right] \leq \mathbb{E}_{t_1, \kappa} \left[J(t_2, \delta, \Theta^{t_2}, K_{t_2}) \right].$$

Together with (1.14) and (1.15), this yields the desired contradiction.

- b) *Monotonicity in δ :* Suppose for a contradiction that U fails to be increasing in δ , i.e., there exist t, x, κ and $0 \leq \delta_1 \leq \delta_2$ such that

$$U(t, \delta_1, x, \kappa) > U(t, \delta_2, x, \kappa). \quad (1.16)$$

Let $\Theta^{\delta_2} \in \mathcal{A}_t(x)$ denote an optimal strategy in the sense that

$$U(t, \delta_2, x, \kappa) = J(t, \delta_2, \Theta^{\delta_2}, \kappa). \quad (1.17)$$

We have seen above that J is increasing in δ such that

$$U(t, \delta_1, x, \kappa) \leq J(t, \delta_1, \Theta^{\delta_2}, \kappa) \leq J(t, \delta_2, \Theta^{\delta_2}, \kappa).$$

Together with (1.16) and (1.17), this yields the desired contradiction.

Monotonicity in x : Suppose for a contradiction that U fails to be increasing in x , i.e., there exist t, δ, κ , $0 \leq x_1 \leq x_2$ such that

$$U(t, \delta, x_1, \kappa) > U(t, \delta, x_2, \kappa). \quad (1.18)$$

Let $\Theta^{x_2} \in \mathcal{A}_t(x_2)$ denote an optimal strategy in the sense that

$$U(t, \delta, x_2, \kappa) = J(t, \delta, \Theta^{x_2}, \kappa). \quad (1.19)$$

Value function	Notation	Boundary conditions
$U(t, \delta, x, \kappa)$		$U(T, \delta, x, \kappa) = \left(\delta + \frac{\kappa}{2}x\right) x$ $U(t, \delta, 0, \kappa) = 0$
$= \delta^2 V(t, y, \kappa)$	$y = \frac{x}{\delta}$ $V(t, y, \kappa) = U(t, 1, y, \kappa)$	$V(T, y, \kappa) = y + \frac{\kappa}{2}y^2$ $V(t, 0, \kappa) = 0$
$= \frac{\delta^2}{\kappa} W(t, z)$	$z = \frac{\kappa x}{\delta}$ $W(t, z) = U(t, 1, z, 1)$	$W(T, z) = z + \frac{1}{2}z^2$ $W(t, 0) = 0$

Table 1.1: Summary of the dimension reduction.

The stopping time

$$\tau := \inf \left\{ s \in [t, \infty) \mid \Theta_s^{x_2} \geq x_1 \right\}$$

is less than or equal to T . Therefore we can define the strategy

$$\Theta_s^{x_1} := \begin{cases} \Theta_s^{x_2} & \text{for } s \in [t, \tau] \\ x_1 & \text{for } s \in (\tau, T] \end{cases}$$

with $\Theta^{x_1} \in \mathcal{A}_t(x_1)$. Due to the construction of τ and Θ^{x_1} , we get

$$\begin{aligned} U(t, \delta, x_1, \kappa) &\leq J(t, \delta, \Theta^{x_1}, \kappa) = \mathbb{E}_{t, \delta, \kappa} \left[\int_{[t, \tau]} \left(D_s + \frac{K_s}{2} \Delta \Theta_s^{x_1} \right) d\Theta_s^{x_1} \right] \\ &\leq J(t, \delta, \Theta^{x_2}, \kappa). \end{aligned}$$

Together with (1.18) and (1.19), this yields the desired contradiction.

Both properties c) and d) can be proved analogously to the monotonicity in δ . \square

1.3 Dimension reduction of the value function

In this section, we prove some scaling properties of the value function which can be ascribed to the block-shape assumption. It is also essential that we assume exponential resilience. That is our method would not work for more general dynamics of D as, e.g., in Predoiu, Shaikhet, and Shreve (2011). The scaling helps us to reduce the dimension of our optimization problem as indicated in Table 1.1.

Lemma 1.3.1. (Optimal strategies scale linearly, Part I).

For all $a \in [0, \infty)$

$$U(t, a\delta, ax, \kappa) = a^2 U(t, \delta, x, \kappa). \quad (1.20)$$

Furthermore, if the strategy $\Theta^* \in \mathcal{A}_t(x)$ is optimal for $U(t, \delta, x, \kappa)$, then $a\Theta^* \in \mathcal{A}_t(ax)$ is optimal for $U(t, a\delta, ax, \kappa)$.

Proof. The assertion is clear for $a = 0$. For any $a \in (0, \infty)$ and $\Theta \in \mathcal{A}_t$, we get from (1.6) and (1.8) that

$$J(t, a\delta, a\Theta, \kappa) = a^2 J(t, \delta, \Theta, \kappa). \quad (1.21)$$

Let $\Theta^* \in \mathcal{A}_t(x)$ be optimal for $U(t, \delta, x, \kappa)$ and $\tilde{\Theta} \in \mathcal{A}_t(ax)$ be optimal for $U(t, a\delta, ax, \kappa)$. If no such optimal strategies exist, the same arguments can be performed with minimizing sequences of strategies. Using (1.21) and the optimality of $\Theta^*, \tilde{\Theta}$, we get

$$\begin{aligned} J(t, a\delta, \tilde{\Theta}, \kappa) &\leq J(t, a\delta, a\Theta^*, \kappa) = a^2 J(t, \delta, \Theta^*, \kappa) \\ &\leq a^2 J\left(t, \delta, \frac{1}{a}\tilde{\Theta}, \kappa\right) = J(t, a\delta, \tilde{\Theta}, \kappa). \end{aligned}$$

That is all inequalities are indeed equalities. Therefore $a\Theta^*$ is optimal for $U(t, a\delta, ax, \kappa)$ and (1.20) holds. \square

For $\delta > 0$, we can take $a = \frac{1}{\delta}$ and apply Lemma 1.3.1 to get

$$U(t, \delta, x, \kappa) = \delta^2 U\left(t, 1, \frac{x}{\delta}, \kappa\right) = \delta^2 V(t, y, \kappa) \quad (1.22)$$

for $y := \frac{x}{\delta}$ and $V(t, y, \kappa) := U(t, 1, y, \kappa)$. That is we are able to reduce our four-dimensional value function (1.9) to a three-dimensional one. The knowledge of $U(t, \delta_{fix}, x, \kappa)$ for some $\delta_{fix} > 0$ or $U(t, \delta, x_{fix}, \kappa)$ for some $x_{fix} > 0$ already gives us the entire value function. Instead of keeping track of the values x and δ separately, only the ratio of them is important. But observe that this does not necessarily mean that V can be interpreted as the value function of a modified optimization problem.

Remark 1.3.2. (Note on $\delta = 0$).

For $x = 0$, the optimization is trivial. For $x > 0$, one could alternatively apply Lemma 1.3.1 to $\tilde{a} = \frac{1}{x}$ such that

$$U(t, \delta, x, \kappa) = x^2 U\left(t, \frac{\delta}{x}, 1, \kappa\right) = x^2 \tilde{V}(t, \tilde{y}, \kappa),$$

for $\tilde{y} := \frac{\delta}{x}$ and $\tilde{V}(t, \tilde{y}, \kappa) := U(t, \tilde{y}, 1, \kappa)$. Anyway, we choose the first alternative with $a = \frac{1}{\delta}$, since the boundary condition $V(t, 0, \kappa) \equiv 0$ is more convenient. It is also more intuitive to think in terms of y instead of \tilde{y} . Only the case $\delta = 0$ would have to be discussed in terms of the second alternative with $y = \frac{x}{\delta} = \infty$ corresponding to $\tilde{y} = \frac{\delta}{x} = 0$. But this would simply be an adjustment of the arguments for the case $\delta > 0$ and it would be notationally awkward. Therefore we do not consider $\delta = 0$ separately.

In some cases, even a reduction to two dimensions is possible.

Lemma 1.3.3. (Optimal strategies scale linearly, Part II).

Assume K to satisfy Assumption GBM. Then for all $a \in (0, \infty)$

$$U\left(t, \delta, ax, \frac{1}{a}\kappa\right) = aU(t, \delta, x, \kappa).$$

Furthermore, if the strategy $\Theta^* \in \mathcal{A}_t(x)$ is optimal for $U(t, \delta, x, \kappa)$, then $a\Theta^* \in \mathcal{A}_t(ax)$ is optimal for $U(t, \delta, ax, \frac{1}{a}\kappa)$.

Proof. For any $a \in (0, \infty)$ and $\Theta \in \mathcal{A}_t$, the deviation process $(D_s)_{s \in [t, T]}$ from (1.8) is identical for Θ and $K_t = \kappa$ compared to $a\Theta$ and $K_t = \frac{1}{a}\kappa$. This is due to *Assumption GBM*. Therefore we get from (1.6) that

$$J\left(t, \delta, a\Theta, \frac{1}{a}\kappa\right) = aJ(t, \delta, \Theta, \kappa).$$

The assertion follows analogously as in the proof of Lemma 1.3.1. \square

Taking (1.22) and applying Lemma 1.3.3 with $a = \kappa$, we derive for $\delta > 0$ that

$$U(t, \delta, x, \kappa) = \frac{\delta^2}{\kappa} U\left(t, 1, \frac{\kappa x}{\delta}, 1\right) = \frac{\delta^2}{\kappa} W(t, z) \quad (1.23)$$

for $z := \frac{\kappa x}{\delta}$ and $W(t, z) := U(t, 1, z, 1)$. That is we are able to reduce our four-dimensional value function (1.9) to a two-dimensional one in case of a *linear* stochastic differential equation for K .

1.4 Hamilton-Jacobi-Bellman equation

For continuous trading time, we can derive the *Hamilton-Jacobi-Bellman equation* (HJB) of the control problem (1.9) using heuristic arguments. To do so, let us assume that U is two times continuously differentiable and that *Assumption Diff* holds. In each situation (t, δ, x, κ) , the large investor has two alternatives: He can either buy the stock or do nothing in order to wait for a more favorable situation.

In the first case, he can trade $\xi > 0$ shares and

$$\begin{aligned} U(t, \delta, x, \kappa) &\leq U(t, \delta + \kappa\xi, x - \xi, \kappa) + \left(\delta + \frac{\kappa}{2}\xi\right)\xi \\ &= U(t, \delta, x, \kappa) + \xi\left(\kappa\partial_\delta U(t, \delta, x, \kappa) - \partial_x U(t, \delta, x, \kappa) + \delta\right) + o(\xi). \end{aligned}$$

We used a Taylor expansion in the last equation and $\partial_\delta U$, $\partial_x U$ are the partial derivatives of U with respect to δ , x , e.g. $\partial_\delta U(t, \delta, x, \kappa) := \frac{\partial U}{\partial \delta}(t, \delta, x, \kappa)$. From this inequality we can conclude that

$$\mathcal{B}^U(U) := \kappa\partial_\delta U - \partial_x U + \delta \geq 0$$

and $\mathcal{B}^U(U) = 0$ if it is optimal to *buy*.

Assume alternatively that the large investor is almost surely not trading on an interval $[t, s]$. By Ito's formula

$$U(t, \delta, x, \kappa) \leq \mathbb{E}_{t, \delta, \kappa} \left[U(s, D_s, x, K_s) \right] = \\ U(t, \delta, x, \kappa) + \mathbb{E}_{t, \delta, \kappa} \left[\int_t^s \left(\partial_t U - \rho_u D_u \partial_\delta U + \mu \partial_\kappa U + \frac{1}{2} \sigma^2 \partial_{\kappa\kappa} U \right) (u, D_u, x, K_u) du \right].$$

This implies

$$\mathcal{W}^U(U) := \partial_t U - \rho_t \delta \partial_\delta U + \mu \partial_\kappa U + \frac{1}{2} \sigma^2 \partial_{\kappa\kappa} U \geq 0$$

and $\mathcal{W}^U(U) = 0$ if it is optimal to *wait* at t .

Altogether, we want to find $U(t, \delta, x, \kappa)$ satisfying boundary conditions (1.12) and the following second order *variational inequality*:

$$\min \{ \mathcal{B}^U(U), \mathcal{W}^U(U) \} = 0. \quad (1.24)$$

That is \mathcal{B}^U denotes the buy and \mathcal{W}^U the wait partial differential equation (PDE) that are valid for the value function U .

Define

$$\mathcal{B}(V) := 1 + 2\kappa V - (1 + \kappa y) \partial_y V, \\ \mathcal{W}(V) := \partial_t V - 2\rho_t V + \rho_t y \partial_y V + \mu \partial_\kappa V + \frac{1}{2} \sigma^2 \partial_{\kappa\kappa} V,$$

and observe that $\mathcal{B}^U(U) = \delta \mathcal{B}(V)$, $\mathcal{W}^U(U) = \delta^2 \mathcal{W}(V)$. That is one can translate (1.24) into a variational inequality for $V(t, y, \kappa)$

$$\min \{ \mathcal{B}(V), \mathcal{W}(V) \} = 0. \quad (1.25)$$

Under *Assumption GBM*, i.e. the illiquidity process is linear, define

$$\mathcal{B}^W(W) := 1 + 2W - (1 + z) \partial_z W, \\ \mathcal{W}^W(W) := \partial_t W - (2\rho_t + \bar{\mu}_t - \bar{\sigma}_t^2) W + (\rho_t + \bar{\mu}_t - \bar{\sigma}_t^2) z \partial_z W + \frac{1}{2} \bar{\sigma}_t^2 z^2 \partial_{zz} W,$$

and observe that $\mathcal{B}^W(W) = \mathcal{B}(V)$, $\mathcal{W}^W(W) = \kappa \mathcal{W}(V)$. That is one can translate (1.25) into a variational inequality for $W(t, z)$

$$\min \{ \mathcal{B}^W(W), \mathcal{W}^W(W) \} = 0. \quad (1.26)$$

Chapter 2

Structural results on optimal execution strategies

Let us give some heuristic arguments on the interpretation of the HJB variational inequality (1.25). It indicates the existence of a *buy* and a *wait* region in our state space. The structure of these sets is both conceptionally and mathematically interesting. For fixed remaining trading time $(T - t)$ and illiquidity κ , one would usually expect the large investor to behave as follows: If there are many shares x left to be bought and the price deviation δ is small, i.e. the ratio $\frac{x}{\delta}$ is large, it is reasonable for the large investor to trade. In the opposite situation, i.e. $\frac{x}{\delta}$ is small, there is no incentive to trade and he, e.g., better waits for a decrease of the price deviation due to resilience. From this reasoning, one would guess that there exists a time- and illiquidity-dependent barrier which divides the buy and the wait region with the buy (wait) region being the values $y = \frac{x}{\delta}$ above (below) this barrier function. Interestingly, it turns out that this natural conjecture does indeed hold true usually, but not always as will be shown in this chapter.

2.1 Introduction to buy and wait region

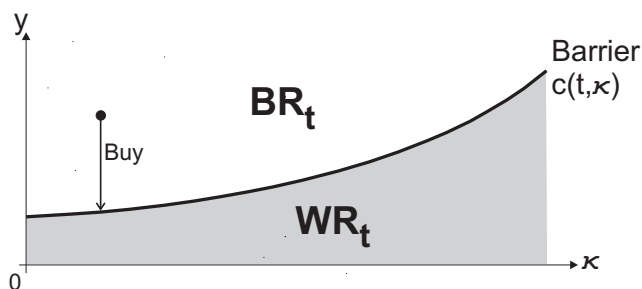


Figure 2.1: Schematic illustration of the buy and wait regions at fixed time t .

We now want to formally introduce the concept of buy and wait region.

Definition 2.1.1. (Buy and wait region).

For given $t \in [0, T)$ and $\kappa \in (0, \infty)$, denote by

$$Br_t(\kappa) := \left\{ y \in (0, \infty) \mid \exists \xi \in (0, y) : U(t, 1, y, \kappa) = U(t, 1 + \kappa\xi, y - \xi, \kappa) + \left(1 + \frac{\kappa}{2}\xi\right) \xi \right\}$$

the *inner buy region* and call the closed sets

$$BR_t(\kappa) := \overline{Br_t(\kappa)}, \quad WR_t(\kappa) := \mathbb{R}_{\geq 0} \setminus Br_t(\kappa)$$

the *buy region* and *wait region*.

For given $t \in [0, T)$, the buy and wait region can then be defined as subsets of \mathbb{R}^2

$$BR_t := \bigcup_{\kappa \in (0, \infty)} \{(y, \kappa) \mid y \in BR_t(\kappa)\}, \quad WR_t := \bigcup_{\kappa \in (0, \infty)} \{(y, \kappa) \mid y \in WR_t(\kappa)\}.$$

In Lemma 2.1.2, we show by a simple argument that it is never optimal to buy all of the remaining shares x before the end of the trading period: It is always better to retain part of the total order to profit from the resilience. In later arguments, we use this fact to conclude that $0 \notin BR_t(\kappa)$ for any $t \in [0, T)$, $\kappa \in (0, \infty)$. If there is not much time left until T and there are only a few shares outstanding, then the large investor does not buy the remainder at once, as would be reasonable from a practical point of view, but spreads it over the remaining time. That is the optimal strategy does not have an intrinsic time horizon.

Lemma 2.1.2. (Trading never completes ahead of time).

For all $t \in [0, T)$, $\delta \in [0, \infty)$ and $x, \kappa \in (0, \infty)$, the value function satisfies

$$U(t, \delta, x, \kappa) < \left(\delta + \frac{\kappa}{2}x\right) x.$$

Proof. For $\epsilon \in [0, x]$, define the admissible strategies $\Theta^\epsilon \in \mathcal{A}_t(x)$ that buy $(x - \epsilon)$ shares at t and ϵ shares at T . The corresponding expected costs are

$$J(t, \delta, \Theta^\epsilon, \kappa) = \left(\delta + \frac{\kappa}{2}(x - \epsilon)\right) (x - \epsilon) + \left((\delta + \kappa[x - \epsilon]) e^{-\int_t^T \rho_s ds} + \frac{\mathbb{E}_{t, \kappa}[K_T]}{2}\right) \epsilon.$$

Clearly,

$$U(t, \delta, x, \kappa) \leq J(t, \delta, \Theta^0, \kappa) = \left(\delta + \frac{\kappa}{2}x\right) x,$$

but we never have equality since

$$\frac{\partial}{\partial \epsilon} J(t, \delta, \Theta^\epsilon, \kappa) \Big|_{\epsilon=0} = - \left(1 - e^{-\int_t^T \rho_s ds}\right) (\kappa x + \delta) < 0.$$

□

The wait region- buy region conjecture can now be formalized as follows.

Definition 2.1.3. (WR-BR structure).

The value function has *WR-BR structure* if there exists a positive *barrier function*

$$c : [0, T) \times (0, \infty) \rightarrow (0, \infty]$$

such that for all $t \in [0, T)$, $\kappa \in (0, \infty)$

$$Br_t(\kappa) = (c(t, \kappa), \infty)$$

with the convention $(\infty, \infty) := \emptyset$.

Thanks to Lemma 2.1.2, it is clear that the barrier c is strictly positive, if it exists. Later on, we will see examples where the barrier attains infinity, i.e., $Br_t(\kappa) = \emptyset$. For $c(t, \kappa) < \infty$, having WR-BR structure means that

$$BR_t(\kappa) \cap WR_t(\kappa) = \{c(t, \kappa)\}.$$

Since trading must be finished off at T , one can include the terminal time T with the convention $c(T, \kappa) \equiv 0$. For the value function U^N in discrete time to have WR-BR structure, we only consider $t \in \{t_0, \dots, t_N\}$ and set $c^N(t, \kappa) = \infty$ for $t \notin \{t_0, \dots, t_N\}$.

Suppose that we work under assumptions on K such that the value function has WR-BR structure and we know the corresponding barrier function. Then the optimal strategy is fully characterized. If the position of the large investor at time t satisfies $\frac{x}{\delta} > c(t, \kappa)$, it is optimal to do the *smallest* discrete trade $\xi \in (0, x)$ such that the new ratio of remaining shares over price deviation $\frac{x-\xi}{\delta+\kappa\xi}$ is not in the interior of the buy region anymore, i.e., the optimal trade then is

$$\xi^* = \frac{x - c(t, \kappa)\delta}{1 + \kappa c(t, \kappa)}$$

which is equivalent to

$$c(t, \kappa) = \frac{x - \xi^*}{\delta + \kappa \xi^*}.$$

Notice that the ratio term $\frac{x-\xi}{\delta+\kappa\xi}$ is strictly decreasing in ξ . Consequently, trades have the effect of reducing the ratio as indicated in Figure 2.1, while the resilience effect increases it. That is one trades just enough shares to keep the ratio y below the barrier. Intuitively, this implies that apart from a possible initial and final impulse trade, optimal buying occurs in infinitesimal amounts provided that c is continuous in t and κ . For diffusive K , this leads to singular optimal controls. Although we do not work explicitly with this interpretation, it is also possible to consider the described issue as a Skorokhod problem with the process $\frac{x-\Theta_t}{D_t}$ being constrained to the set WR_t with reflection at the barrier due to trading. All in all, it is desirable to prove WR-BR structure of the value function and to compute the corresponding barrier function. Let us have a look at the following proposition which formally states these ideas.

Proposition 2.1.4. (WR-BR structure equivalent to trading towards the barrier).
Assume that for each (t, δ, x, κ) there exists a unique optimal strategy to problem (1.9)

$$(\Theta_s^*(t, \delta, x, \kappa))_{s \in [t, T]} \in \mathcal{A}_t(x).$$

Then the following statements are equivalent.

(a) The value function has WR-BR structure.

(b) There exists $c : [0, T] \times (0, \infty) \rightarrow (0, \infty]$ such that for all (t, δ, x, κ)

$$\Delta \Theta_t^*(t, \delta, x, \kappa) = \max \left\{ 0, \frac{x - c(t, \kappa)\delta}{1 + \kappa c(t, \kappa)} \right\}. \quad (2.1)$$

In particular, $\Delta \Theta_t^*(t, \delta, x, \kappa)$ is continuous in δ and x .

(c) For all (t, δ, κ) , the function $x \mapsto \Delta \Theta_t^*(t, \delta, x, \kappa)$ is increasing.

In the sequel and especially for the proof of Proposition 2.1.4, we use the following simple observation.

Lemma 2.1.5. (Splitting argument).

Doing two separate trades $\xi_\alpha, \xi_\beta > 0$ at the same time $t \in [0, T]$ has the same effect as trading at once $\xi := \xi_\alpha + \xi_\beta$, i.e., this incurs the same impact costs and impact D_{t+} .

Proof. The impact costs are in both cases

$$\begin{aligned} \left(D_t + \frac{K_t}{2} \xi \right) \xi &= D_t (\xi_\alpha + \xi_\beta) + \frac{K_t}{2} (\xi_\alpha^2 + 2\xi_\alpha \xi_\beta + \xi_\beta^2) \\ &= \left(D_t + \frac{K_t}{2} \xi_\alpha \right) \xi_\alpha + \left(D_t + K_t \xi_\alpha + \frac{K_t}{2} \xi_\beta \right) \xi_\beta \end{aligned}$$

and the impact $D_{t+} = D_t + K_t (\xi_\alpha + \xi_\beta)$ after the trade is equal as well. \square

Proof of Proposition 2.1.4. Firstly, we prove the equivalence of (a) and (b). Statement (c) follows immediately from (b). We conclude by showing that (c) implies (b). Recall the dimension reduction from Lemma 1.3.1 which yields

$$\Delta \Theta_t^*(t, \delta, x, \kappa) = \delta \Delta \Theta_t^* \left(t, 1, \frac{x}{\delta}, \kappa \right).$$

That is we only need to discuss the case $\delta = 1$. Fix arbitrary $t \in [0, T]$, $\kappa \in (0, \infty)$.

(a) \Rightarrow (b) The assertion holds for $x = 0$. Assume $x \in (0, c(t, \kappa)]$. Then WR-BR structure implies that for all $\xi \in (0, x)$

$$U(t, 1, x, \kappa) < U(t, 1 + \kappa\xi, x - \xi, \kappa) + \left(1 + \frac{\kappa}{2}\xi\right) \xi.$$

Therefore it cannot be optimal to trade immediately at time t .

Assume $c(t, \kappa) < \infty$ and $x \in (c(t, \kappa), \infty)$. Then WR-BR structure implies that there exists $\tilde{\xi} \in (0, x)$ such that

$$U(t, 1, x, \kappa) = U\left(t, 1 + \kappa\tilde{\xi}, x - \tilde{\xi}, \kappa\right) + \left(1 + \frac{\kappa}{2}\tilde{\xi}\right) \tilde{\xi}.$$

Due to the uniqueness of the optimal strategy, we get

$$\Delta\Theta_t^*(t, 1, x, \kappa) = \tilde{\xi} + \Delta\Theta_t^*\left(t, 1 + \kappa\tilde{\xi}, x - \tilde{\xi}, \kappa\right) > 0.$$

For $\tilde{\xi} < \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)}$, we have $\frac{x - \tilde{\xi}}{1 + \kappa\tilde{\xi}} > c(t, \kappa)$ and thus

$$\Delta\Theta_t^*\left(t, 1 + \kappa\tilde{\xi}, x - \tilde{\xi}, \kappa\right) > 0.$$

Consequently, $\Delta\Theta_t^*(t, 1, x, \kappa) \geq \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)}$ and due to Lemma 2.1.5

$$\Delta\Theta_t^*(t, 1, x, \kappa) = \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)} + \Delta\Theta_t^*\left(t, 1 + \kappa \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)}, x - \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)}, \kappa\right).$$

Observe that the second summand equals zero because

$$\frac{x - \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)}}{1 + \kappa \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)}} = c(t, \kappa).$$

(b) \Rightarrow (a) Assume $x \in (0, c(t, \kappa)]$. Then (2.1) implies $\Delta\Theta_t^*(t, 1, x, \kappa) = 0$. Together with the uniqueness of the optimal strategy we can therefore conclude that $x \notin Br_t(\kappa)$, since for all $\xi \in (0, x)$

$$U(t, 1, x, \kappa) < U(t, 1 + \kappa\xi, x - \xi, \kappa) + \left(1 + \frac{\kappa}{2}\xi\right) \xi.$$

Assume $c(t, \kappa) < \infty$ and $x \in (c(t, \kappa), \infty)$. Then (2.1) implies

$$\Delta\Theta_t^*(t, 1, x, \kappa) \in (0, x).$$

The optimality of Θ^* leads to the conclusion $x \in Br_t(\kappa)$ since

$$\begin{aligned} U(t, 1, x, \kappa) &= U\left(t, 1 + \kappa\Delta\Theta_t^*(t, 1, x, \kappa), x - \Delta\Theta_t^*(t, 1, x, \kappa), \kappa\right) \\ &\quad + \left(1 + \frac{\kappa}{2}\Delta\Theta_t^*(t, 1, x, \kappa)\right) \Delta\Theta_t^*(t, 1, x, \kappa). \end{aligned}$$

(c) \Rightarrow (b) Define

$$c(t, \kappa) := \inf \{x \in (0, \infty) \mid \Delta\Theta_t^*(t, 1, x, \kappa) > 0\}.$$

We are done for $c(t, \kappa) = \infty$. For $c(t, \kappa) < \infty$, the definition of $c(t, \kappa)$ guarantees $\Delta\Theta_t^* = 0$ for all $x < c(t, \kappa)$ and Property (c) implies $\Delta\Theta_t^* > 0$ for all $x > c(t, \kappa)$. Suppose for a contradiction that

$$\Delta\Theta_t^*(t, 1, c(t, \kappa), \kappa) > 0.$$

Due to the uniqueness and the splitting argument, we then have

$$\epsilon \in [0, \Delta\Theta_t^*(t, 1, c(t, \kappa), \kappa)) \text{ with}$$

$$\Delta\Theta_t^*(t, 1, c(t, \kappa), \kappa) = \epsilon + \Delta\Theta_t^*(t, 1 + \kappa\epsilon, c(t, \kappa) - \epsilon, \kappa) = \epsilon < \Delta\Theta_t^*(t, 1, c(t, \kappa), \kappa).$$

Therefore $\Delta\Theta_t^* = 0$ for all $x \leq c(t, \kappa)$.

We still need to prove $\Delta\Theta_t^* = \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)}$ for $x > c(t, \kappa)$. Assume $\Delta\Theta_t^* > \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)}$. Once more, we make use of the uniqueness and the splitting argument in order to get a contradiction

$$\begin{aligned} \Delta\Theta_t^* &= \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)} + \Delta\Theta_t^*\left(t, 1 + \kappa \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)}, x - \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)}, \kappa\right) \\ &= \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)} < \Delta\Theta_t^*. \end{aligned}$$

Assume $\Delta\Theta_t^* < \frac{x - c(t, \kappa)}{1 + \kappa c(t, \kappa)}$. That is $\frac{x - \Delta\Theta_t^*}{1 + \kappa \Delta\Theta_t^*} > c(t, \kappa)$ such that we also get a contradiction

$$\Delta\Theta_t^* = \Delta\Theta_t^* + \Delta\Theta_t^*(t, 1 + \kappa \Delta\Theta_t^*, x - \Delta\Theta_t^*, \kappa) > \Delta\Theta_t^*.$$

□

In discrete trading time, one can also show that not only the initial trade of the optimal strategy $\Delta\Theta_{t_n}^*(t_n, \delta, x, \kappa)$ has the form (2.1), but also all the consequent optimal trades $\Delta\Theta_{t_j}^*(t_n, \delta, x, \kappa)$ for $j = n + 1, \dots, N$.

Proposition 2.1.6.

(Discrete time: WR-BR structure equivalent to trading towards the barrier).

Assume that for each (t_n, δ, x, κ) there exists a unique optimal strategy to problem (1.10)

$$(\Theta_s^*(t_n, \delta, x, \kappa))_{s \in [t_n, T]} \in \mathcal{A}_{t_n}^N(x).$$

Then the following statements are equivalent.

(a) The value function has WR-BR structure.

(b) There exists $c^N : [0, T] \times (0, \infty) \rightarrow (0, \infty]$ such that for all (t_n, δ, x, κ) and $j = n, \dots, N$

$$\Delta \Theta_{t_j}^* (t_n, \delta, x, \kappa) = \max \left\{ 0, \frac{x - \Theta_{t_j}^* (t_n, \delta, x, \kappa) - c^N (t_j, K_{t_j}) D_{t_j}^* (t_n, \delta, x, \kappa)}{1 + K_{t_j} c^N (t_j, K_{t_j})} \right\}. \quad (2.2)$$

Proof. Recalling Proposition 2.1.4, it is clear that (b) implies (a). We show the reverse direction by induction. For $j - n = 0$, equation (2.2) follows from Proposition 2.1.4. Consider the induction step. For all n , we want to show (2.2) for $j - n$ when it already holds for $j - n - 1$. Applying the uniqueness of the optimal strategy and the dynamic programming principle in the first and third equation and the induction hypothesis in the second one, we get

$$\begin{aligned} & \Delta \Theta_{t_j}^* (t_n, \delta, x, \kappa) \\ &= \Delta \Theta_{t_j}^* \left[t_{n+1}, (\delta + \kappa \Delta \Theta_{t_n}^* (t_n, \delta, x, \kappa)) e^{-\int_{t_n}^{t_{n+1}} \rho_s ds}, x - \Delta \Theta_{t_n}^* (t_n, \delta, x, \kappa), K_{t_{n+1}} \right] \\ &= \max \left\{ 0, \frac{x - \Delta \Theta_{t_n}^* (t_n, \delta, x, \kappa) - \Theta_{t_j}^* [\dots] - c^N (t_j, K_{t_j}) D_{t_j}^* [\dots]}{1 + K_{t_j} c^N (t_j, K_{t_j})} \right\} \\ &= \max \left\{ 0, \frac{x - \Theta_{t_j}^* (t_n, \delta, x, \kappa) - c^N (t_j, K_{t_j}) D_{t_j}^* (t_n, \delta, x, \kappa)}{1 + K_{t_j} c^N (t_j, K_{t_j})} \right\}. \end{aligned}$$

Here the term [...] stands for the content of the square brackets one line above. \square

We have seen in Proposition 2.1.4 and 2.1.6 that the idea of WR-BR structure is closely related to optimal trading being increasing in the number of shares still to be bought. Therefore, we conjectured at the beginning of this project that WR-BR structure should hold under general assumptions on K . But it turns out that this is not necessarily the case. It is indeed possible to construct examples, which we call *WR-BR-WR examples*, where

$$y_1 \in BR_t(\kappa) \text{ and } y_2 \in WR_t(\kappa) \text{ for } y_1 < y_2.$$

This makes it even more important to have a closer look at the concept of WR-BR structure. Thus, we discuss the presented control problem for different special cases of the K dynamics. For each case, we want to answer the following questions:

- Under which conditions does WR-BR structure hold?
- If we have WR-BR structure, what properties does the barrier have?
- Can we compute the value function and the optimal strategy numerically?
- Is it possible to state an explicit WR-BR-WR example?

Table 2.1 gives an overview under which assumptions on K we are going to prove WR-BR structure. It also lists the corresponding sections.

	WR-BR structure	WR-BR-WR example
Discrete time	2.2.1, <i>Assumption Const</i> 2.2.2, <i>Assumption Determ</i> 2.3, <i>Assumption SpecialGBM</i> 2.4.2, <i>Assumption SpecialDiff</i> 2.4.3, <i>Assumption SpecialCIR</i>	2.5.1, Binomial model 2.5.3, GBM 2.5.2, CIR
Continuous time	2.2.1, <i>Assumption Const</i> 2.2.3, <i>Assumption CtsDeterm</i> 2.4.2, <i>Assumption SpecialDiff</i> 2.4.3, <i>Assumption SpecialCIR</i>	2.5.4, Binomial model

Table 2.1: Outlook of the following WR-BR structural results in discrete and continuous trading time.

2.2 Deterministic price impact

Here, we go beyond the existing literature and consider deterministic K . The subsequent sections of this chapter contain the stochastic case. The purpose of a deterministic specification of the illiquidity process is the inclusion of seasonalities in the order book dynamics. For example, the traded volume is often observed to be U-shaped during one trading day, see e.g. Lorenz and Osterrieder (2009). Malo and Pennanen (2010) and Cont, Kukanov, and Stoikov (2010) empirically find that the averaged intraday pattern of the order book depth is increasing. This would correspond to a decreasing K . In view of Proposition 1.1.1, all optimal strategies will automatically be deterministic in this section.

In Subsection 2.2.1, we state the closed-form solution of our optimization problem for constant K . In Subsection 2.2.2, we start by proving that we always have WR-BR structure in the deterministic case when trading is allowed in discrete time only. In continuous time, we need the deterministic illiquidity process to be continuous in order to guarantee the existence of optimal strategies, see Subsection 2.2.3. Apart from this continuity of K , the WR-BR result carries over from discrete to continuous time without any further assumptions. In some cases, we can use the Euler-Lagrange formalism in Subsection 2.2.4 to compute closed-form optimal strategies for deterministic K .

2.2.1 Constant price impact

As a first example of our model, let us discuss the case where resilience and limit order book stay constant over time, i.e., $\frac{\partial}{\partial s}\rho_s \equiv 0$ and K satisfies *Assumption Const*. This corresponds to the model suggested by Obizhaeva and Wang (2006). They show in particular that a unique optimal strategy exists which is increasing in the number of shares still to be bought. This guarantees WR-BR structure via Proposition 2.1.4. As

an illustration, we are going to calculate the barrier function and the full solution to the variational inequality. The procedure will be as follows: We recall the optimal strategy for $\delta = 0$. From this strategy, we calculate the barrier function $c(t, \kappa)$. With this barrier function at hand, we can then compute optimal strategies for $\delta > 0$. We double-check that these strategies are indeed optimal by showing that the corresponding costs satisfy the variational inequality. In particular, the whole procedure will explicitly point out how one gets the optimal strategy for a given barrier and vice versa.

Proposition 2.2.1. (Constant liquidity, discrete time).

Under Assumption Const and for $(N + 1)$ equidistant trading times $t_n = n\tau$, $\tau := \frac{T}{N}$, the optimal strategy $\Theta^N \in \mathcal{A}_0^N(x)$ for $\delta = 0$ is given by

$$\begin{aligned} \xi_0^N = \xi_N^N &= \frac{x}{(N-1)(1-e^{-\rho\tau})+2}, \\ \xi_1^N = \dots = \xi_{N-1}^N &= \frac{x-2\xi_0^N}{N-1} = \frac{(1-e^{-\rho\tau})x}{(N-1)(1-e^{-\rho\tau})+2}. \end{aligned}$$

For $n = 1, \dots, N$, the deviation is $D_{t_n}^N = \kappa\xi_0^N e^{-\rho\tau}$ and we get the barrier

$$c^N(t_n, \kappa) := \frac{x - \Theta_{t_n+}^N}{D_{t_n+}} = \frac{\tilde{c}^N(t_n)}{\kappa}, \quad \tilde{c}^N(t_n) := e^{-\rho\tau} + \frac{1-e^{-\rho\tau}}{\tau}(T-t_n).$$

The corresponding optimal costs are

$$U^N(0, 0, x, \kappa) = \frac{1}{2} \left(1 + N \frac{1-e^{-\rho\tau}}{1+e^{-\rho\tau}} \right)^{-1} \kappa x^2.$$

In the more general case $\delta \geq 0$, $n \in \{0, \dots, N\}$ and $\frac{\kappa x}{\delta} \geq \tilde{c}^N(t_n)$, we get

$$U^N(t_n, \delta, x, \kappa) = \frac{1}{2} \left(1 + (N-n) \frac{1-e^{-\rho\tau}}{1+e^{-\rho\tau}} \right)^{-1} \left[\kappa x^2 + 2x\delta - (N-n) \frac{1-e^{-\rho\tau}}{1+e^{-\rho\tau}} \frac{\delta^2}{\kappa} \right].$$

The optimal strategy ξ^N in this explicit form is given in Alfonsi, Fruth, and Schied (2010) and D^N , c^N as well as $U^N(0, 0, x, \kappa)$ can be calculated directly from ξ^N . Having calculated the barrier, it is also easy to compute the value function in more general cases. The optimal strategy for $\delta = 0$ does not depend on κ . It is linear in x and therefore has constant relative trading speed.

Proposition 2.2.2. (Constant liquidity, continuous time).

Under Assumption Const, the optimal strategy $\Theta \in \mathcal{A}_0(x)$ for $\delta = 0$ is given by

$$\Delta\Theta_0 = \Delta\Theta_T = \frac{x}{\rho T + 2}, \quad d\Theta_t = \frac{x - 2\Delta\Theta_0}{T} = \frac{\rho x}{\rho T + 2}. \quad (2.3)$$

For $t \in (0, T]$, the deviation is $D_t \equiv \kappa\Delta\Theta_0$ and we get the barrier

$$c(t, \kappa) := \frac{x - \Theta_{t+}}{D_{t+}} = \frac{\tilde{c}(t)}{\kappa}, \quad \tilde{c}(t) := 1 + \rho(T-t).$$

The corresponding optimal costs are

$$U(0, 0, x, \kappa) = (\rho T + 2)^{-1} \kappa x^2.$$

In the more general case $\delta > 0$, $t \in [0, T]$ and $\frac{\kappa x}{\delta} \geq \tilde{c}(t)$ we get

$$U(t, \delta, x, \kappa) = (\rho(T - t) + 2)^{-1} \left[\kappa x^2 + 2x\delta - \frac{\rho}{2}(T - t) \frac{\delta^2}{\kappa} \right].$$

This optimal strategy for $\delta = 0$ in continuous time, i.e., $\Theta_0 = 0$, $\Theta_s = \frac{\rho t + 1}{\rho T + 2} x$ for $s \in [0, T]$, is taken from Obizhaeva and Wang (2006). As already mentioned above, it is linear in the number of shares. Notice also that \tilde{c}^N is increasing in N with

$$\lim_{N \rightarrow \infty} \tilde{c}^N(t) \nearrow \tilde{c}(t).$$

The barrier $c(t, \kappa)$ behaves like $\frac{1}{\kappa}$ and is a decreasing straight line in the time dimension, i.e., the shorter the time to maturity, the bigger is the buy region. Moreover, it is characteristic that the large investor exactly purchases the shares that replenish the order book due to the resilience effect. Thus the deviation process under the optimal strategy is constant— the large investor always trades at this optimal price deviation. Also observe that the value function has the form

$$\alpha(t)\kappa x^2 + \beta(t)x\delta + \gamma(t) \frac{\delta^2}{\kappa}$$

and in particular is increasing and concave in κ due to γ being negative.

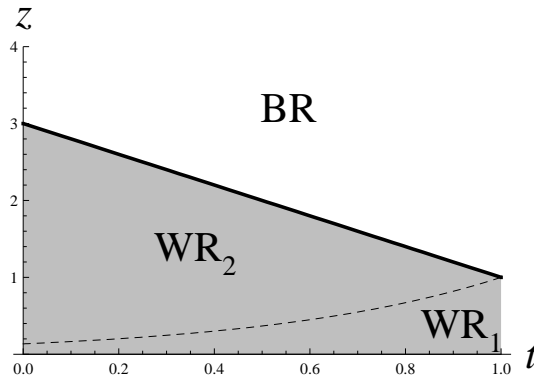


Figure 2.2: Illustration of the buy and wait regions for constant κ and $T = 1, \rho = 2$.

Neither Obizhaeva and Wang (2006) nor Alfonsi, Fruth, and Schied (2010) consider the case of strictly positive price deviations due to previous trades. But now that we have calculated the barrier function, we can deduce the optimal strategy and the value function for $\delta > 0$. For $\frac{x}{\delta}$ above the barrier $c(t, \kappa)$ (buy region), the investor does a discrete trade $\Delta\Theta_t > 0$ to reach the barrier, then trades at constant rate $d\Theta_s$ to stay

on the barrier and finally buys the remaining shares at T . For $\frac{x}{\delta}$ below the barrier (wait region), he does not trade until

$$\tilde{t}(t, z) := \inf \{s > t \mid z e^{\rho(s-t)} = 1 + \rho(T - s)\}, \quad (2.4)$$

which is the time when the resilience has brought the position z up to the barrier. Starting at \tilde{t} , the investor trades at constant rate to stay on the barrier and finishes the order at T . But for low values of z , $\tilde{t} \geq T$ which means that it is not profitable to trade before T . Thus the whole order is unwound at T , i.e., $\Delta\Theta_T = x$. See Figure 2.2 for an illustration of the different trading regions. This optimal strategy is explicitly stated in Proposition 2.2.3. One calculates the corresponding costs as

$$\begin{aligned} W^{BR}(t, z) &= \frac{z^2 + 2z - \frac{1}{2}\rho(T-t)}{\tilde{c}(t) + 1}, \\ W^{WR_2}(t, z) &= \frac{1}{2}z^2 + [1 - \rho(T - \tilde{t})]e^{-\rho(\tilde{t}-t)}z + \frac{\rho^2}{2}(T - \tilde{t})^2 e^{-2\rho(\tilde{t}-t)}, \\ W^{WR_1}(t, z) &= \frac{1}{2}z^2 + e^{-\rho(T-t)}z. \end{aligned}$$

Proposition 2.2.3. (Constant liquidity, continuous time, supplement).

Under Assumption Const, we state the optimal strategy $\Theta \in \mathcal{A}_t(x)$ for $\delta > 0$:

- In the buy region, i.e., $\frac{\kappa x}{\delta} \geq \tilde{c}(t)$, the initial trade, the constant trading rate for $s \in (t, T)$ and the final trade are

$$\Delta\Theta_t = \frac{x - \frac{\delta}{\kappa}\tilde{c}(t)}{\tilde{c}(t) + 1}, \quad d\Theta_s \equiv \rho \frac{x + \frac{\delta}{\kappa}}{\tilde{c}(t) + 1} ds, \quad \Delta\Theta_T = \frac{x + \frac{\delta}{\kappa}}{\tilde{c}(t) + 1}. \quad (2.5)$$

- In the upper wait region, i.e., $e^{-\rho(T-t)} < \frac{\kappa x}{\delta} < \tilde{c}(t)$, we only trade on $[\tilde{t}, T]$ with \tilde{t} from (2.4) and

$$\Delta\Theta_t = 0, \quad d\Theta_s = \rho \frac{\delta e^{-\rho(\tilde{t}-t)}}{\kappa} \mathbb{I}_{\{s \in [\tilde{t}, T]\}} ds, \quad \Delta\Theta_T = x - \rho \frac{(T - \tilde{t})\delta e^{-\rho(\tilde{t}-t)}}{\kappa}.$$

- In the lower wait region, i.e., $\frac{\kappa x}{\delta} \leq e^{-\rho(T-t)}$, it is optimal to trade the entire order at T

$$\Delta\Theta_t = 0, \quad d\Theta_s \equiv 0, \quad \Delta\Theta_T = x.$$

The corresponding optimal costs are $U(t, \delta, x, \kappa) = \frac{\delta^2}{\kappa} W(t, z)$ with

$$W(t, z) := \left\{ \begin{array}{ll} W^{BR}(t, z) & \text{if } z \in [\tilde{c}(t), \infty) \\ W^{WR_2}(t, z) & \text{if } z \in (e^{-\rho(T-t)}, \tilde{c}(t)) \\ W^{WR_1}(t, z) & \text{if } z \in [0, e^{-\rho(T-t)}] \end{array} \right\}.$$

The function W is a classical solution of the first order variational inequality (1.26) for $\rho_t \equiv \rho$ and $\bar{\mu}_t = \bar{\sigma}_t \equiv 0$. It is C^2 everywhere and C^∞ for points (t, z) with $z \in [0, \infty) \setminus \{\tilde{c}(t), e^{-\rho(T-t)}\}$.

Proof. One can check the regularity directly and further calculations show

$$\begin{aligned} \mathcal{B}^W(W^{BR}) &= \mathcal{W}^W(W^{WR_2}) = \mathcal{W}^W(W^{WR_1}) = 0 \quad \text{and} \\ \mathcal{W}^W(W^{BR}), \mathcal{B}^W(W^{WR_2}), \mathcal{B}^W(W^{WR_1}) &\geq 0. \end{aligned}$$

□

We go ahead by looking at more general deterministic profiles for K .

2.2.2 Discrete time wait and buy region structure

In this section, we work under *Assumption Determ.* Let us consider discrete trading time and, slightly abusing notation, positive numbers $K_n := K_{t_n}$ for $n = 0, \dots, N$. Define $D_n := D_{t_n}$ and

$$U^N(t_n, \delta, x) := U^N(t_n, \delta, x, K_n) = \inf_{\substack{\xi_j \in [0, x] \\ \sum \xi_j = x}} \sum_{j=n}^N \left(D_j + \frac{K_j}{2} \xi_j \right) \xi_j.$$

That is $D_n = \delta$, $D_{j+1} = (D_j + K_j \xi_j) a_j$ with the resilience multipliers

$$a_j := \exp \left(- \int_{t_j}^{t_{j+1}} \rho_s ds \right). \quad (2.6)$$

Recall the dimension reduction from Table 1.1

$$U^N(t_n, \delta, x) = \delta^2 V^N \left(t_n, \frac{x}{\delta} \right) \quad \text{with} \quad V^N(t_n, y) := U^N(t_n, 1, y).$$

We are now going to show by backward induction that the value function $V^N(t_n, \cdot)$ is continuously differentiable and piecewise quadratic with coefficients satisfying inequalities (2.7) below. We can exploit these properties in order to prove WR-BR structure.

Proposition 2.2.4. (Deterministic, discrete time case: WR-BR structure).

Under Assumption Determ, the discrete time value function has WR-BR structure, a unique optimal strategy exists and for $n = 0, \dots, N$, $V^N(t_n, \cdot) : [0, \infty) \rightarrow [0, \infty)$ has the following properties.

- (i) *It is continuously differentiable.*
- (ii) *It is piecewise quadratic, i.e., there exists $M \in \mathbb{N}$, constants $(\alpha_i, \beta_i, \gamma_i)_{i=1, \dots, M}$ and $0 < y_1 < y_2 < \dots < y_M = \infty$ such that*

$$V^N(t_n, y) = \alpha_{m(y)} y^2 + \beta_{m(y)} y + \gamma_{m(y)}$$

for the index function $m : [0, \infty) \rightarrow \{1, \dots, M\}$ with $m(y) := \min\{i | y \leq y_i\}$.

(iii) The coefficients $(\alpha_i, \beta_i, \gamma_i)_{i=1, \dots, M}$ from (ii) satisfy the **inequalities**

$$\begin{aligned} \alpha_i, \beta_i &> 0, \\ 4\alpha_i\gamma_i + \beta_i - \beta_i^2 &\geq 0, \\ y_{i-1}\beta_i + 2\gamma_i &\geq 0. \end{aligned} \tag{2.7}$$

The fact that the value function is piecewise quadratic comes about since we assume the price impact D to be multiplied by the trade and the impact itself is linear in the trade.

We prove the proposition by backward induction. To do so, the following lemmata are essential in the induction step. They deal with the value function at times t_n and t_{n+1} . We know that these functions are linked by the dynamic programming equation (1.13). In the proof of Lemma 2.2.5, we see that this connection between $U^N(t_n, \cdot, \cdot)$ and $U^N(t_{n+1}, \cdot, \cdot)$ can be rewritten such that the minimization is not taken over the trade ξ , but equivalently over the new ratio

$$\eta := \frac{y - \xi}{1 + K_n \xi}.$$

This minimization over the new ratio in turn relates to the minimization of the auxiliary function $L^N(t_n, \cdot)$ defined via $V^N(t_{n+1}, \cdot)$ in (2.10) below. In the sequel, it will be essential in several arguments. Since $L^N(t_n, \cdot)$ is strictly decreasing on $[0, c^*)$ and strictly increasing on (c^*, ∞) according to Lemma 2.2.6 a), we find the unique optimal strategy by choosing $\eta = c^* \wedge y$. That is we trade to the barrier if $y > c^*$ and do not trade otherwise. Applying Proposition 2.1.4 proves WR-BR structure.

Lemma 2.2.5. (Optimal trade and optimal barrier equation).

Under Assumption Determ, the discrete time value function satisfies

$$U^N(t_n, \delta, x) = \min_{\xi \in [0, x]} \left\{ \left(\delta + \frac{K_n}{2} \xi \right) \xi + U^N(t_{n+1}, (\delta + K_n \xi) a_n, x - \xi) \right\}, \tag{2.8}$$

$$U^N(t_n, \delta, x) = \min_{\eta \in [0, \frac{x}{\delta}]} \frac{1}{2K_n} [(\delta + K_n x)^2 L^N(t_n, \eta) - \delta^2], \tag{2.9}$$

$$L^N(t_n, y) := \frac{1 + 2K_n a_n^2 V^N(t_{n+1}, y a_n^{-1})}{(1 + K_n y)^2}. \tag{2.10}$$

Proof. Applying the dynamic programming principle from (1.13) yields

$$\begin{aligned} U^N(t_n, \delta, x) &= \min_{\xi \in [0, x]} \left\{ \left(\delta + \frac{K_n}{2} \xi \right) \xi + U^N(t_{n+1}, (\delta + K_n \xi) a_n, x - \xi) \right\} \\ &= \min_{\xi \in [0, x]} \left\{ \left(\delta + \frac{K_n}{2} \xi \right) \xi + (\delta + K_n \xi)^2 a_n^2 V^N \left(t_{n+1}, \frac{x - \xi}{\delta + K_n \xi} a_n^{-1} \right) \right\}. \end{aligned}$$

Instead of focusing on the optimal trade, one can alternatively look for the optimal new ratio of remaining shares over price deviation. To do so, we define the decreasing new

ratio $\eta(\xi) := \frac{x-\xi}{\delta+K_n\xi}$ that results from a trade ξ . If we are not trading, the ratio stays the same, i.e., $\eta(0) = \frac{x}{\delta}$ and if we trade everything at once, it gets zero, i.e., $\eta(x) = 0$. Due to $\xi(\eta) = \frac{x-\eta\delta}{1+K_n\eta}$ we can rewrite (2.8) as

$$U^N(t_n, \delta, x) = \min_{\eta \in [0, \frac{x}{\delta}]} \frac{1}{2K_n} [(\delta + K_n x)^2 L^N(t_n, \eta) - \delta^2].$$

□

Lemma 2.2.6. *Let $b > 1, \kappa > 0$ and $v : [0, \infty) \rightarrow [0, \infty)$ satisfy (i), (ii), (iii) given in Proposition 2.2.4.*

a) *There exists $c^* \in [0, \infty]$ such that*

$$L(y) := \frac{1 + 2\kappa b^{-2}v(yb)}{(1 + \kappa y)^2}$$

is strictly decreasing for $y < c^$ and strictly increasing for $y > c^*$.*

b) *The function*

$$\tilde{v}(y) := \begin{cases} \frac{1}{2\kappa} [(1 + \kappa y)^2 L(c^*) - 1] & \text{if } y > c^* \\ b^{-2}v(yb) & \text{otherwise} \end{cases}$$

again satisfies (i), (ii), (iii) with possibly different coefficients.

Proof of Proposition 2.2.4. Notice that $V^N(t_N, y) = (1 + \frac{K_N}{2}y)y$ fulfills (i), (ii), (iii) with $M = 1, \alpha_1 = \frac{K_N}{2}, \beta_1 = 1, \gamma_1 = 0$. Let us consider the induction step from t_{n+1} to t_n . We are going to use Lemma 2.2.5 and 2.2.6 for $b = a_n^{-1}, \kappa = K_n, v = V^N(t_{n+1}, \cdot)$ and $L = L^N(t_n, \cdot)$. Take $c_n := c^*$ from Lemma 2.2.6 a). Since the ratio η after the trade cannot be larger than the ratio $\frac{x}{\delta}$ before the trade, we get that

$$\eta^* := \operatorname{argmin}_{\eta \in [0, \frac{x}{\delta}]} \frac{1}{2K_n} [(\delta + K_n x)^2 L^N(t_n, \eta) - \delta^2] = \min \left\{ \frac{x}{\delta}, c_n \right\}$$

and accordingly

$$\xi^* := \xi(\eta^*) = \max \left\{ 0, \frac{x - c_n \delta}{1 + K_n c_n} \right\}.$$

That is we have a unique optimal strategy and WR-BR structure follows from Proposition 2.1.4. The barrier c_n has to be strictly positive due to Lemma 2.1.2. Plugging ξ^* into (2.8) and doing the dimension reduction from Table 1.1 gives $V^N(t_n, y) = \tilde{v}(y)$. Therefore Lemma 2.2.6 b) concludes the induction step. □

Proof of Lemma 2.2.6. a) The function L is continuously differentiable with

$$L'(y) = \frac{2\kappa}{(1 + \kappa y)^3} l(y), \tag{2.11}$$

$$l(y) := y(2\alpha_{m(yb)} - \kappa\beta_{m(yb)}b^{-1}) + (\beta_{m(yb)}b^{-1} - 2\kappa\gamma_{m(yb)}b^{-2} - 1).$$

First of all, we show that there is no interval where L is constant. Assume there would be an interval where l is zero, i.e., there exists $i \in \{1, \dots, M\}$ such that $(2\alpha_i - \kappa\beta_i b^{-1}) = 0$ and $(\beta_i b^{-1} - 2\kappa\gamma_i b^{-2} - 1) = 0$. Solving these equations for α respectively γ yields

$$4\alpha_i\gamma_i + b\beta_i - \beta_i^2 = 0.$$

This is a contradiction to (2.7).

Let us assume $l(\check{y}) > 0$ for some $\check{y} \in [0, \infty)$ with $j := m(\check{y}b)$. We are done if we can conclude $l(\hat{y}) > 0$ for all $\hat{y} \in [\check{y}, \infty)$. Because of the continuity of l , it is sufficient to show that L keeps increasing on $[\check{y}, y_j]$, i.e., we need to show $l(\hat{y}) > 0$ for all $\hat{y} \in [\check{y}, y_j]$. Due to the form of l , this is guaranteed when $2\alpha_j - \kappa\beta_j b^{-1} > 0$. Let us suppose that this term would be nonpositive which is equivalent to $2\alpha_j\beta_j^{-1}b \leq \kappa$. Together with the inequalities from (2.7) one gets

$$\begin{aligned} bl(\check{y}) &= -\kappa b^{-1}(\check{y}b\beta_j + 2\gamma_j) + (2\check{y}b\alpha_j + \beta_j - b) \\ &\leq -2\alpha_j\beta_j^{-1}(\check{y}b\beta_j + 2\gamma_j) + (2\check{y}b\alpha_j + \beta_j - b) \\ &= -\frac{1}{\beta_j}(4\alpha_j\gamma_j + \beta_j b - \beta_j^2) < 0. \end{aligned}$$

This is a contradiction to $l(\check{y}) > 0$.

b) If c^* is finite, the function \tilde{v} is continuously differentiable in c^* since a brief calculation shows that $\tilde{v}'(c^* -) = \tilde{v}'(c^* +)$ is equivalent to $l(c^*) = 0$.

$$\tilde{v}(y) = \tilde{\alpha}_{\tilde{m}(y)}y^2 + \tilde{\beta}_{\tilde{m}(y)}y + \tilde{\gamma}_{\tilde{m}(y)}$$

Directly from its definition, one sees that \tilde{v} is piecewise quadratic with $\tilde{M} = 1 + m(c^*b)$, $\tilde{y}_{\tilde{M}-1} := c^*$, $\tilde{y}_i := y_i b^{-1}$ for $i = 1, \dots, \tilde{M} - 2$ and

$$\begin{aligned} \tilde{\alpha}_{\tilde{M}} &= \frac{\kappa}{2}L(c^*) > 0, \quad \tilde{\beta}_{\tilde{M}} = L(c^*) > 0, \quad \tilde{\gamma}_{\tilde{M}} = \frac{L(c^*) - 1}{2\kappa}, \\ \tilde{\alpha}_i &= \alpha_i > 0, \quad \tilde{\beta}_i = b^{-1}\beta_i > 0, \quad \tilde{\gamma}_i = b^{-2}\gamma_i \quad \text{for } i = 1, \dots, \tilde{M} - 1. \end{aligned} \tag{2.12}$$

We therefore get

$$4\tilde{\alpha}_i\tilde{\gamma}_i + \tilde{\beta}_i - \tilde{\beta}_i^2 = \begin{cases} 0 & \text{if } i = \tilde{M} \\ b^{-2}(4\alpha_i\gamma_i + b\beta_i - \beta_i^2) & \text{otherwise} \end{cases} \geq 0.$$

It remains to show that \tilde{v} also inherits the last inequality in (2.7) from v . For $y \leq c^*$,

$$y\tilde{\beta}_{\tilde{m}(y)} + 2\tilde{\gamma}_{\tilde{m}(y)} = b^{-2}(yb\beta_{m(yb)} + 2\gamma_{m(yb)}) \geq 0.$$

Due to \tilde{v} being continuously differentiable in c^* , we get

$$\begin{aligned} \tilde{\alpha}_{\tilde{M}}(c^*)^2 + \tilde{\beta}_{\tilde{M}}c^* + \tilde{\gamma}_{\tilde{M}} &= \tilde{\alpha}_{\tilde{M}-1}(c^*)^2 + \tilde{\beta}_{\tilde{M}-1}c^* + \tilde{\gamma}_{\tilde{M}-1}, \\ 2\tilde{\alpha}_{\tilde{M}}c^* + \tilde{\beta}_{\tilde{M}} &= 2\tilde{\alpha}_{\tilde{M}-1}c^* + \tilde{\beta}_{\tilde{M}-1}. \end{aligned}$$

Taking two times the first equation and subtracting c^* times the second equation yields

$$c^* \tilde{\beta}_{\tilde{M}} + 2\tilde{\gamma}_{\tilde{M}} = c^* \tilde{\beta}_{\tilde{M}-1} + 2\tilde{\gamma}_{\tilde{M}-1}.$$

Since we already know that the right hand side is positive, we finally get $y\tilde{\beta}_{\tilde{M}} + 2\tilde{\gamma}_{\tilde{M}} \geq 0$ for all $y > c^*$. \square

The proof of Proposition 2.2.4 not only establishes the existence of a unique barrier $(c(t_n))_{n=0,\dots,N}$, but also directly states how to compute this barrier numerically. Let us state the corresponding backward recursive algorithm.

Initialize value function $V^N(t_N, y) = (1 + \frac{K_N}{2}y) y$
 For $n = N - 1, \dots, 0$
 Set $L^N(t_n, y) := \frac{1+2K_n a_n^2 V^N(t_{n+1}, y a_n^{-1})}{(1+K_n y)^2}$
 Compute $c(t_n) := c_n := \operatorname{argmin}_{y \geq 0} L^N(t_n, y)$
 Set $V^N(t_n, y) := \left\{ \begin{array}{ll} \frac{1}{2K_n} [(1 + K_n y)^2 L^N(t_n, c_n) - 1] & \text{if } y > c_n \\ a_n^2 V^N(t_{n+1}, y a_n^{-1}) & \text{otherwise} \end{array} \right\}$

2.2.3 Continuous time wait and buy region structure

Existence of an optimal strategy

Let us now show how the WR-BR result from Subsection 2.2.2 carries over to continuous time. It is crucial to first discuss the existence of optimal strategies in continuous time for deterministic K . We do so by working with Helly's compactness theorem. Example 2.2.7 and 2.2.8 motivate the fact that we will work under *Assumption CtsDeterm* in this subsection.

Example 2.2.7. (Illiquidity as Dirichlet function: No optimal strategy exists).

Take $T = 1$, constant resilience speed and the following artificial example

$$K(t) = \left\{ \begin{array}{ll} 1 & \text{for } t \text{ rational} \\ 2 & \text{for } t \text{ irrational} \end{array} \right\}.$$

Let us show that the corresponding value function $U(0, 0, x)$ is equal to $U^{OW}(0, 0, x, 1)$ from Proposition 2.2.2 for constant K . Observe that $U \geq U^{OW}$ due to $K_t \geq 1$ for all t . We get $U \leq U^{OW}$ from the following argumentation. Let $\mathcal{T}^N = \{t_0^N, \dots, t_N^N\}$ be an equidistant partition of $[0, T]$, i.e., $t_n^N = \frac{n}{N}$. Call Θ^N the optimal strategy for constant K and trading allowed on \mathcal{T}^N only. Then $K_{t_n^N} = 1$ for all $N \in \mathbb{N}, n \in \{0, \dots, N\}$ such that

$$U(0, 0, x) \leq \lim_{N \rightarrow \infty} J(0, 0, \Theta^N) = U^{OW}(0, 0, x, 1).$$

However, among all $\Theta \in \mathcal{A}_0(x)$ there is no strategy that attains $U^{OW}(0, 0, x, 1)$. Since the set of rational numbers is countable, no measure without atoms will charge it with positive mass. We can therefore write

$$\begin{aligned}
J(0, 0, \Theta) &= \int_{[0, T]} \left(D_t + \frac{K_t}{2} \Delta \Theta_t \right) d\Theta_t \\
&= \int_{[0, T]} \left(\int_{[0, t]} K_s e^{-\rho(t-s)} d\Theta_s \right) d\Theta_t^c + \sum_{t \in [0, T]} \left[D_t + \frac{K_t}{2} \Delta \Theta_t \right] \Delta \Theta_t \\
&= \int_{[0, T]} \left(2 \int_{[0, t]} e^{-\rho(t-s)} d\Theta_s^c + \sum_{s \in [0, t)} K_s \Delta \Theta_s e^{-\rho(t-s)} \right) d\Theta_t^c \\
&\quad + \sum_{t \in [0, T]} \left[D_t + \frac{K_t}{2} \Delta \Theta_t \right] \Delta \Theta_t \\
&\geq \int_{[0, T]} \left(\int_{[0, t]} e^{-\rho(t-s)} d\Theta_s^c \right) d\Theta_t^c + J^{OW}(0, 0, \Theta, 1). \tag{2.13}
\end{aligned}$$

If $\Theta_T^c > 0$, the term $J(0, 0, \Theta)$ must be strictly larger than $J^{OW}(0, 0, \Theta, 1)$. Suppose for a contradiction that there exists an optimal strategy $\Theta^* \in \mathcal{A}_0(x)$ such that

$$U(0, 0, x) = J(0, 0, \Theta^*) \geq J^{OW}(0, 0, \Theta^*, 1) \geq U^{OW}(0, 0, x, 1). \tag{2.14}$$

We know from above that the left-hand and right-hand side coincide. Due to the uniqueness of the optimal strategy Θ^{OW} in the case of constant K , we get $\Theta^* = \Theta^{OW}$. Therefore we deduce from (2.13) that

$$J(0, 0, \Theta^*) > J^{OW}(0, 0, \Theta^*, 1).$$

This is a contradiction to (2.14). Therefore there exists no strategy that attains the value function.

Example 2.2.8. (Illiquidity with a jump: Costs are not continuous in the strategy).

Assume $K_t = \check{\kappa}$ and $K_s = \hat{\kappa}$ for all $s \in (t, T]$ with $0 < \check{\kappa} < \hat{\kappa}$. That is K has a jump at initial time. The sequence of strategies

$$\Theta_s^n := \left\{ \begin{array}{ll} 0 & \text{if } s \leq t + \frac{1}{n} \\ x & \text{otherwise} \end{array} \right\}$$

converges weakly to $\tilde{\Theta}_s := x \mathbb{I}_{(t, T]}(s)$, but we have no convergence of the costs, since

$$J(t, \delta, \tilde{\Theta}) = \left(\delta + \frac{\check{\kappa}}{2} x \right) x \neq \lim_{n \rightarrow \infty} J(t, \delta, \Theta^n) = \left(\delta + \frac{\hat{\kappa}}{2} x \right) x.$$

Hence, this example illustrates that K being continuous is a sensible condition for the assertion in Proposition 2.2.9.

We will tackle the existence of an optimal strategy for deterministic, continuous K in continuous time in Proposition 2.2.12 by use of the auxiliary result Proposition 2.2.9. We can then prove in Theorem 2.2.15 that the value function has WR-BR structure.

Proposition 2.2.9. (Costs are continuous in the strategy, K continuous).

Under Assumption CtsDeterm, let $\tilde{\Theta}, (\Theta^n)$ be deterministic strategies in $\mathcal{A}_t(x)$ with $\Theta^n \xrightarrow{w} \tilde{\Theta}$, i.e., $\lim_{n \rightarrow \infty} \Theta_s^n = \tilde{\Theta}_s$ for every point $s \in [t, T]$ of continuity of $\tilde{\Theta}$. Then

$$\left| J(t, \delta, \tilde{\Theta}) - J(t, \delta, \Theta^n) \right| \xrightarrow{n \rightarrow \infty} 0.$$

Notice that $\tilde{\Theta}_{T+} = \Theta_{T+}^n = x$, so we do not need to assume convergence at $T+$. In order to prove Proposition 2.2.9, we first show that the convergence of the price impact processes follows from the weak convergence of the corresponding strategies. We then conclude that a version of Proposition 2.2.9 for absolutely continuous K holds. This finally leads to the desired version for continuous K .

Lemma 2.2.10. (Price impact process is continuous in the strategy).

Under Assumption CtsDeterm, let $\tilde{\Theta}, (\Theta^n)$ be deterministic strategies in $\mathcal{A}_t(x)$ with $\Theta^n \xrightarrow{w} \tilde{\Theta}$. Then $\lim_{n \rightarrow \infty} D_s^n = \tilde{D}_s$ for $s = T+$ and for every point $s \in [t, T]$ of continuity of $\tilde{\Theta}$.

Proof. Recall equation (1.8)

$$D_s = \int_t^s K_u e^{-\int_u^s \rho_r dr} d\Theta_u + \delta e^{-\int_t^s \rho_u du},$$

which holds for $s = T+$ and $s \in [t, T]$. Due to the weak convergence and the integrand being continuous in u , the assertion follows for $s = T+$. The weak convergence also tells us that for all $s \in [t, T]$ with $\Delta\tilde{\Theta}_s = 0$ and $f_s(u) := K_u e^{-\int_u^s \rho_r dr} I_{[t,s]}(u)$

$$\begin{aligned} D_s^n &= \int_t^{T+} f_s(u) d\Theta_u^n + \delta e^{-\int_t^s \rho_u du} \xrightarrow{n \rightarrow \infty} \\ &\int_t^{T+} f_s(u) d\tilde{\Theta}_u + \delta e^{-\int_t^s \rho_u du} = \tilde{D}_s. \end{aligned}$$

□

Lemma 2.2.11. (Costs are continuous in the strategy, K absolutely continuous).

Under Assumption CtsDeterm, let K be absolutely continuous and $\tilde{\Theta}, (\Theta^n)$ be deterministic strategies in $\mathcal{A}_t(x)$ with $\Theta^n \xrightarrow{w} \tilde{\Theta}$. Then

$$\left| J(t, \delta, \tilde{\Theta}) - J(t, \delta, \Theta^n) \right| \xrightarrow{n \rightarrow \infty} 0.$$

Proof. We can write $K_s = K_t + \int_t^s \mu_u du$. Let us show that

$$J(t, \delta, \Theta) = \frac{1}{2} \left[\frac{D_{T+}^2}{K_T} - \frac{\delta^2}{K_t} + \int_{[t, T]} \left(\frac{2\rho_s}{K_s} + \frac{\mu_s}{K_s^2} \right) D_s^2 ds \right]. \quad (2.15)$$

The assertion then follows from Lemma 2.2.10. In order to get (2.15), we apply

$$d\Theta_s = \frac{dD_s + \rho_s D_s ds}{K_s}, \quad \Delta\Theta_s = \frac{\Delta D_s}{K_s}.$$

This implies

$$\begin{aligned} J(t, \delta, \Theta) &= \int_{[t, T]} \left(D_s + \frac{K_s}{2} \Delta\Theta_s \right) d\Theta_s \\ &= \int_{[t, T]} \frac{D_s + \frac{1}{2} \Delta D_s}{K_s} dD_s + \int_{[t, T]} \frac{\rho_s D_s^2}{K_s} ds + \int_{[t, T]} \frac{\frac{1}{2} \Delta D_s \rho_s D_s}{K_s} ds. \end{aligned}$$

In this expression, the last term is zero since D has only countably many jumps. Using integration by parts for càglàd processes and $d\left(\frac{D_s}{K_s}\right) = \frac{1}{K_s} dD_s + D_s d\left(\frac{1}{K_s}\right)$, we can write

$$\int_{[t, T]} \frac{D_s}{K_s} dD_s = \frac{1}{2} \left[\frac{D_{T+}^2}{K_T} - \frac{\delta^2}{K_t} - \int_{[t, T]} D_s^2 d\left(\frac{1}{K_s}\right) - \sum_{s \in [t, T]} \frac{(\Delta D_s)^2}{K_s} \right].$$

Plugging in $d\left(\frac{1}{K_s}\right) = -\frac{\mu_s}{K_s^2} ds$ yields (2.15). \square

Proof of Proposition 2.2.9. We use a proof by contradiction. Suppose there exists a subsequence $(n_j) \subset \mathbb{N}$ such that

$$\lim_{j \rightarrow \infty} \int_{[t, T]} \left(D_s^{n_j} + \frac{K_s}{2} \Delta\Theta_s^{n_j} \right) d\Theta_s^{n_j} \neq \int_{[t, T]} \left(\tilde{D}_s + \frac{K_s}{2} \Delta\tilde{\Theta}_s \right) d\tilde{\Theta}_s$$

and the limit in this equation exists. Without loss of generality assume

$$\lim_{j \rightarrow \infty} \int_{[t, T]} \left(D_s^{n_j} + \frac{K_s}{2} \Delta\Theta_s^{n_j} \right) d\Theta_s^{n_j} < \int_{[t, T]} \left(\tilde{D}_s + \frac{K_s}{2} \Delta\tilde{\Theta}_s \right) d\tilde{\Theta}_s. \quad (2.16)$$

We now want to bring Lemma 2.2.11 into play. For $\epsilon > 0$, we denote by K^ϵ an absolutely continuous modification of K such that $\max_{s \in [t, T]} |K_s^\epsilon - K_s| \leq \epsilon$. For $\Theta \in \mathcal{A}_t(x)$

$$\begin{aligned} & \left| \int_{[t, T]} \left(D_s^\epsilon + \frac{K_s^\epsilon}{2} \Delta\Theta_s \right) d\Theta_s - \int_{[t, T]} \left(D_s + \frac{K_s}{2} \Delta\Theta_s \right) d\Theta_s \right| \\ & \leq \int_{[t, T]} \left(|D_s^\epsilon - D_s| + \frac{1}{2} |K_s^\epsilon - K_s| \Delta\Theta_s \right) d\Theta_s \leq \frac{3}{2} x^2 \epsilon. \end{aligned}$$

We therefore get from (2.16) that there exists $\epsilon > 0$ such that

$$\limsup_{j \rightarrow \infty} \int_{[t, T]} \left(D_s^{n_j, \epsilon} + \frac{K_s^\epsilon}{2} \Delta \Theta_s^{n_j} \right) d\Theta_s^{n_j} < \int_{[t, T]} \left(\tilde{D}_s^\epsilon + \frac{K_s^\epsilon}{2} \Delta \tilde{\Theta}_s \right) d\tilde{\Theta}_s.$$

This is a contradiction to Lemma 2.2.11. □

Proposition 2.2.12. (Deterministic, continuous time case: Existence).

Under Assumption CtsDeterm, there exists an optimal strategy $\Theta^ \in \mathcal{A}_t(x)$, i.e.*

$$J(t, \delta, \Theta^*) = \inf_{\Theta \in \mathcal{A}_t(x)} J(t, \delta, \Theta).$$

Proof. Let $(\Theta^n) \subset \mathcal{A}_t(x)$ be a deterministic, minimizing sequence. Due to the monotonicity of the considered strategies, we can use *Helly's Theorem* in the form of Theorem 2, §2, Chapter III of Shiryaev (1995), which also holds for left-continuous processes and on $(0, T]$ instead of $(-\infty, \infty)$. It guarantees the existence of a deterministic $\tilde{\Theta} \in \mathcal{A}_t(x)$ and a subsequence $(n_j) \subset \mathbb{N}$ such that (Θ^{n_j}) converges weakly to $\tilde{\Theta}$. Thanks to Proposition 2.2.9, we can conclude that

$$U(t, \delta, x) = \lim_{j \rightarrow \infty} J(t, \delta, \Theta^{n_j}) = J(t, \delta, \tilde{\Theta}).$$

□

Remark 2.2.13. This existence proof does not work for stochastic K , since the subsequence (n_j) would be scenario dependent.

Corollary 2.2.14. (Approximation by continuous strategies).

Define the restricted strategy set with impulse trades at initial and capital time only

$$\mathcal{A}_t^c(x) := \{\Theta \in \mathcal{A}_t(x) \mid \Theta \text{ continuous on } (t, T)\}.$$

Under Assumption CtsDeterm,

$$\inf_{\Theta \in \mathcal{A}_t(x)} J(t, \delta, \Theta) = \inf_{\Theta \in \mathcal{A}_t^c(x)} J(t, \delta, \Theta).$$

Proof. Proposition 2.2.12 guarantees the existence of a deterministic optimal strategy $\Theta^* \in \mathcal{A}_t(x)$ with countably many jump times $t < s_0 < s_1 < \dots < T$ with $\Delta \Theta_{s_j}^* > 0$. For $\epsilon > 0$, let $\Theta^\epsilon \in \mathcal{A}_t^c(x)$ be an approximation of Θ^* : Set $\Theta_s^\epsilon := \Theta_s^*$, but on $(s_0, s_0 + \epsilon)$

$$\Theta_s^\epsilon := \Theta_{s_0}^* + \frac{\Theta_{s_0+\epsilon}^* - \Theta_{s_0}^*}{\epsilon} (s - s_0).$$

Proceed analogously for the next jump time larger or equal to $s_0 + \epsilon$, and so on. Then $\Theta^\epsilon \xrightarrow{w} \Theta^*$ for $\epsilon \searrow 0$. Hence, with Proposition 2.2.9

$$\inf_{\Theta \in \mathcal{A}_t(x)} J(t, \delta, \Theta) = J(t, \delta, \Theta^*) = \lim_{\epsilon \searrow 0} J(t, \delta, \Theta^\epsilon) \geq \inf_{\Theta \in \mathcal{A}_t^c(x)} J(t, \delta, \Theta).$$

□

WR-BR structure

For K being deterministic and continuous, we have now finalized the existence result. Let us go ahead by showing that not only the value function in discrete, but also in continuous time has WR-BR structure. Apart from the continuity of K , we do not impose any assumptions. Hence, we can not work with the HJB equation and the corresponding viscosity solutions. Instead, we are going to deduce the structural result for our free boundary problem by transferring our discrete time result.

Theorem 2.2.15. (Deterministic, continuous time case: WR-BR structure).
Under Assumption CtsDeterm, the value function has WR-BR structure.

We now sketch the main idea of the proof. First, we show that the discrete time value function converges to the continuous time value function. Second, we prove certain convexity properties of the discrete time auxiliary function $L^N(0, \cdot)$. Finally, these properties yield uniform convergence of $L^N(0, \cdot)$ in the number of trading instances such that all nice properties from discrete time carry over to continuous time.

Without loss of generality, we set $t = 0$.

Lemma 2.2.16. (Discrete time converges to continuous time value function).
Consider an equidistant time grid with $N_j := 2^j$ trading instances. Under Assumption CtsDeterm,

$$\lim_{j \rightarrow \infty} V^{N_j}(0, y) = V(0, y).$$

Lemma 2.2.17. (No buy region within the wait region).

Recall that $c^N(0)$ denotes the barrier at $t = 0$, when we consider $N + 1$ trading instances. Under Assumption Determ, at least one of the following two statements is true:

- *For all $N \in \mathbb{N}$, the function $y \mapsto L^N(0, y)$ given in (2.10) is convex on $[0, c^N(0))$.*
- *The continuous time buy region is simply $Br_0 = \emptyset$, i.e. $c(0) = \infty$.*

Lemma 2.2.18. (L^N gets flat on the buy region).

Under Assumption CtsDeterm, consider an equidistant time grid with $N + 1$ entries and assume $\bar{c}(0) := \sup_{N \in \mathbb{N}} c^N(0) < \infty$. Then for each $y \in [\bar{c}(0), \infty)$

$$\lim_{N \rightarrow \infty} [L^N(0, y) - L^N(0, c^N(0))] = 0.$$

Proof of Theorem 2.2.15. As stated in Lemma 2.2.5 for discrete time, the continuous time value function satisfies both the optimal trade and optimal barrier equation:

$$\begin{aligned} V(0, y) &= \min_{\xi \in [0, y]} \left\{ \left(1 + \frac{K_0}{2}\xi\right) \xi + (1 + K_0\xi)^2 V\left(0, \frac{y - \xi}{1 + K_0\xi}\right) \right\}, \\ V(0, y) &= \min_{\eta \in [0, y]} \frac{1}{2K_0} [(1 + K_0\eta)^2 L(0, \eta) - 1], \\ L(0, y) &:= L(y) := \frac{1 + 2K_0V(0, y)}{(1 + K_0y)^2}. \end{aligned} \tag{2.17}$$

The function L is positive with $L(0) = 1$. Define

$$c(0) := \inf \left\{ y \geq 0 \mid \exists \hat{y} \in (y, \infty) : L(y) \leq L(\hat{y}) \right\}.$$

Then $c(0) \in (0, \infty]$ due to Lemma 2.1.2 and L is strictly decreasing on $[0, c(0))$. We only need to show that for finite $c(0)$, L is constant on $[c(0), \infty)$. One can then conclude that for all $y > c(0)$ and $\eta \in (c(0), y)$, equivalently $\xi := \frac{y-\eta}{1+K_0\eta} \in \left(0, \frac{y-c(0)}{1+K_0c(0)}\right)$, we have

$$V(0, y) = \frac{1}{2K_0} \left[(1 + K_0y)^2 L(0, \eta) - 1 \right] = \left(1 + \frac{K_0}{2} \xi \right) \xi + (1 + K_0\xi)^2 V \left(0, \frac{y - \xi}{1 + K_0\xi} \right).$$

Moreover, for all $y \leq c(0)$ and $\eta \in [0, y)$, equivalently $\xi \in (0, y]$, $V(0, y)$ is strictly smaller than the right-hand side. That is $Br_0 = (c(0), \infty)$ as desired.

Hence, it only remains to prove that for finite $c(0)$, L is constant on $[c(0), \infty)$. To do so, we show that the limit of $L^{N_j}(0, \cdot)$ has this property and that this limit coincides with L . For $N_j = 2^j$ and the corresponding equidistant time grid, such a limit exists at least pointwise since for each fixed y

$$j \mapsto V^{N_j} \left(\frac{T}{N_j}, y e^{\int_0^{\frac{T}{N_j}} \rho_s ds} \right)$$

is decreasing in j due to Lemma 1.2.2 and stays positive. We call the limit

$$\tilde{L}(y) := \lim_{j \rightarrow \infty} L^{N_j}(0, y)$$

and define $\tilde{c}(0)$ analogously to $c(0)$. That is \tilde{L} must be strictly decreasing up to $\tilde{c}(0)$. Due to Lemma 2.2.17 and 2.2.18, \tilde{L} must also be convex up to $\tilde{c}(0)$, and for finite $\tilde{c}(0)$, it is constant on $[\tilde{c}(0), \infty)$ as desired. In particular, \tilde{L} is non-increasing.

Let us show $L = \tilde{L}$ to complete the proof. According to Lemma 2.2.16 and 2.2.5,

$$V(0, y) = \lim_{j \rightarrow \infty} V^{N_j}(0, y) = \lim_{j \rightarrow \infty} \min_{\eta \in [0, y]} \frac{1}{2K_0} \left[(1 + K_0y)^2 L^{N_j}(0, \eta) - 1 \right]. \quad (2.18)$$

We would like to interchange the limit and the minimum such that

$$V(0, y) = \min_{\eta \in [0, y]} \frac{1}{2K_0} \left[(1 + K_0y)^2 \tilde{L}(\eta) - 1 \right]. \quad (2.19)$$

In order to achieve this, we first do some preparations and then differentiate two cases.

Preparations: Define a modification of $L^{N_j}(0, \cdot)$ by setting it constant on $[c^{N_j}(0), \infty)$

$$\check{L}^{N_j}(0, y) := \left\{ \begin{array}{ll} L^{N_j}(0, y) & \text{for } y \leq c^{N_j}(0) \\ L^{N_j}(0, c^{N_j}(0)) & \text{otherwise} \end{array} \right\}.$$

Our discrete time result guarantees that this modified function only takes values in $[0, 1]$ and due to Lemma 2.2.17, it can be assumed to be convex. Otherwise, the second point in Lemma 2.2.17 would apply and we would be finished with the whole proof straight away. Thanks to Lemma 2.2.18, \check{L}^{N_j} has the same limit as L^{N_j} for each fixed y , i.e.

$$\lim_{j \rightarrow \infty} \check{L}^{N_j}(0, y) = \check{L}(y).$$

Furthermore, the modified function attains the same minimum as $L^N(0, \cdot)$ itself, in the sense that we can replace (2.18) by

$$V(0, y) = \frac{1}{2K_0} \left[(1 + K_0 y)^2 \left[\lim_{j \rightarrow \infty} \min_{\eta \in [0, y]} \check{L}^{N_j}(0, \eta) \right] - 1 \right]. \quad (2.20)$$

Case 1: Assume $\limsup_{j \rightarrow \infty} c^{N_j}(0) > 0$, i.e. there exists a subsequence (\check{N}_j) of (N_j) such that $\check{c}(0) := \lim_{j \rightarrow \infty} c^{\check{N}_j}(0) \in (0, \infty]$. Then we claim that

$$\lim_{j \rightarrow \infty} \min_{\eta \in [0, y]} \check{L}^{N_j}(0, \eta) = \lim_{j \rightarrow \infty} \min_{\eta \in [\frac{\check{c}(0)}{2} \wedge y, y]} \check{L}^{\check{N}_j}(0, \eta) = \min_{\eta \in [\frac{\check{c}(0)}{2} \wedge y, y]} \check{L}(\eta) = \min_{\eta \in [0, y]} \check{L}(\eta). \quad (2.21)$$

Indeed, Rockafellar (1997), Theorem II.10.8 guarantees uniform convergence of $\check{L}^{N_j}(0, \cdot)$ on the closed bounded set $[\frac{\check{c}(0)}{2} \wedge y, y] \subset (0, \infty)$, since this pointwise converging sequence consists of finite, convex functions on $(0, \infty)$. In the last equation of (2.21), we exploit the fact that \check{L} is non-increasing on $[0, \frac{\check{c}(0)}{2}]$. With (2.20) and (2.21) we get (2.19).

Case 2: Assume $\limsup_{j \rightarrow \infty} c^{N_j}(0) = \lim_{j \rightarrow \infty} c^{N_j}(0) = 0$, i.e. for each $y \in (0, \infty)$ there exists $\tilde{j} \in \mathbb{N}$ such that $c^{N_j}(0) < y$ for all $j \geq \tilde{j}$. Due to $\check{L}^{N_j}(0, \cdot)$ being constant on $[c^{N_j}(0), \infty)$ and \check{L} being non-increasing, we conclude

$$\lim_{j \rightarrow \infty} \min_{\eta \in [0, y]} \check{L}^{N_j}(0, \eta) = \check{L}(y) = \min_{\eta \in [0, y]} \check{L}(\eta).$$

After convincing ourselves that (2.19) holds, we now want to exploit this fact to deduce $L = \check{L}$. We get $\check{L} \geq L$ due to

$$\lim_{j \rightarrow \infty} V^{N_j} \left(\frac{T}{N_j}, y e^{\int_0^{\frac{T}{N_j}} \rho_s ds} \right) \geq \lim_{j \rightarrow \infty} V^{N_j}(0, y) = V(0, y).$$

With (2.17) and (2.19), it follows that \check{L} and L must be equal. In particular, $\tilde{c}(0) = c(0) > 0$ such that *Case 2* can be excluded a posteriori. We can now conclude that for finite $c(0)$ not only \check{L} , but also L is constant on $[c(0), \infty)$. \square

Proof of Lemma 2.2.16. Thanks to Proposition 2.2.12, there exists a continuous time optimal strategy $\Theta^* \in \mathcal{A}_0(y)$. Approximate it via step functions $\Theta^j \in \mathcal{A}_0^{N_j}(y)$. This will be explained in more detail in Lemma 2.4.7 below. Then

$$V(0, y) = J(0, 1, \Theta^*) = \lim_{j \rightarrow \infty} J(0, 1, \Theta^j) \geq \lim_{j \rightarrow \infty} V^{N_j}(0, y).$$

The inequality $V(0, y) \leq \lim_{j \rightarrow \infty} V^{N_j}(0, y)$ is immediate. \square

Proof of Lemma 2.2.17. Fix $N \in \mathbb{N}$. Recall that the definition of $L^N(0, \cdot)$ from (2.10) contains $V^N(t_1, \cdot)$ which is continuously differentiable and piecewise quadratic with coefficients $(\alpha_i, \beta_i, \gamma_i)$. Analogously to (2.11), it turns out that

$$\begin{aligned} \frac{\partial}{\partial y} L^N(0, y) &= \frac{2K_0}{(1 + K_0 y)^3} \left[y \left(2\alpha_m \left(y e^{\int_0^{t_1} \rho_s ds} \right) - K_0 \beta_m \left(y e^{\int_0^{t_1} \rho_s ds} \right) e^{-\int_0^{t_1} \rho_s ds} \right) \right. \\ &\quad \left. + \left(\beta_m \left(y e^{\int_0^{t_1} \rho_s ds} \right) e^{-\int_0^{t_1} \rho_s ds} + 2K_0 \gamma_m \left(y e^{\int_0^{t_1} \rho_s ds} \right) e^{-2\int_0^{t_1} \rho_s ds} - 1 \right) \right]. \end{aligned}$$

We differentiate two cases. First assume that all i satisfy $(2\alpha_i - K_0 \beta_i e^{-\int_0^{t_1} \rho_s ds}) \geq 0$. Then $\frac{\partial}{\partial y} L^N(0, \cdot)$ must be increasing on $[0, c^N(0))$ as desired, since $L^N(0, \cdot)$ is decreasing on this interval as we know from the discrete time result Lemma 2.2.6.

Assume to the contrary that there exists i such that $(2\alpha_i - K_0 \beta_i e^{-\int_0^{t_1} \rho_s ds}) < 0$. Recall how α_i and β_i are actually computed in the backward induction of Proposition 2.2.4. In each induction step, Lemma 2.2.6 is used and the coefficients $\tilde{\alpha}_{\tilde{M}}, \tilde{\beta}_{\tilde{M}}$ get updated in (2.12). It gets clear that there exists $n \in \{1, \dots, N\}$ such that

$$2\alpha_i - K_0 \beta_i e^{-\int_0^{t_1} \rho_s ds} = \left(K_{t_n} - K_0 e^{-\int_0^{t_n} \rho_s ds} \right) L^N(t_n, c^N(t_n)).$$

We get the resilience multiplier $e^{-\int_0^{t_n} \rho_s ds}$ thanks to the adjustment $\tilde{\beta}_i = a^{-1} \beta_i$ from the second line of (2.12). Due to L^N being positive, it follows that

$$K_{t_n} < K_0 e^{-\int_0^{t_n} \rho_s ds}.$$

That is for this deterministic choice of K , it cannot be optimal to trade at $t = 0$. Hence, the buy region at $t = 0$ is the empty set for both discrete and continuous time. \square

Proof of Lemma 2.2.18. Due to $c^N(0)$ being the minimum of $L^N(0, \cdot)$, we know that

$$0 \leq \liminf_{N \rightarrow \infty} [L^N(0, y) - L^N(0, c^N(0))] \leq \limsup_{N \rightarrow \infty} [L^N(0, y) - L^N(0, c^N(0))].$$

That is we only need to show that the upper limit is less or equal to zero. Suppose for a contradiction that there exists $y \in [\bar{c}(0), \infty)$, $\epsilon > 0$ such that for all $\tilde{N} \in \mathbb{N}$ there is an $N \geq \tilde{N}$ with

$$L^N(0, y) - L^N(0, c^N(0)) \geq \epsilon.$$

Then for $t_1 = \frac{T}{N}$

$$\begin{aligned} V^N(0, y) &= \frac{1}{2K_0} [(1 + K_0 y)^2 L^N(0, c^N(0)) - 1] \\ &\leq \frac{1}{2K_0} [(1 + K_0 y)^2 \{L^N(0, y) - \epsilon\} - 1] \\ &= V^N(t_1, y e^{\int_0^{t_1} \rho_s ds}) e^{-2\int_0^{t_1} \rho_s ds} - \epsilon \frac{(1 + K_0 y)^2}{2K_0}. \end{aligned}$$

In the last equation, we plugged in the definition of $L^N(0, y)$ from (2.10). We can analogously do these calculations for the optimal trade instead of optimal barrier equation

$$\begin{aligned}
V^N(0, y) &= \left(1 + \frac{K_0}{2} \frac{y - c^N(0)}{1 + K_0 c^N(0)}\right) \frac{y - c^N(0)}{1 + K_0 c^N(0)} \\
&\quad + \left(1 + K_0 \frac{y - c^N(0)}{1 + K_0 c^N(0)}\right)^2 V^N\left(t_1, c^N(0) e^{\int_0^{t_1} \rho_s ds}\right) e^{-2 \int_0^{t_1} \rho_s ds} \\
&\leq V^N\left(t_1, y e^{\int_0^{t_1} \rho_s ds}\right) e^{-2 \int_0^{t_1} \rho_s ds} - \epsilon \frac{(1 + K_0 y)^2}{2K_0} \\
&\leq \left(1 + \frac{K_{t_1}}{2} \frac{y - c^N(0)}{e^{-\int_0^{t_1} \rho_s ds} + K_{t_1} c^N(0)}\right) \frac{y - c^N(0)}{e^{-\int_0^{t_1} \rho_s ds} + K_{t_1} c^N(0)} e^{-2 \int_0^{t_1} \rho_s ds} \\
&\quad + \left(1 + K_{t_1} \frac{y - c^N(0)}{e^{-\int_0^{t_1} \rho_s ds} + K_{t_1} c^N(0)}\right)^2 V^N\left(t_1, c^N(0) e^{\int_0^{t_1} \rho_s ds}\right) e^{-2 \int_0^{t_1} \rho_s ds} \\
&\quad - \epsilon \frac{(1 + K_0 y)^2}{2K_0}.
\end{aligned} \tag{2.22}$$

We now explain the auxiliary calculation that is involved in the last inequality. The aim is to estimate $V^N\left(t_1, y e^{\int_0^{t_1} \rho_s ds}\right)$ in terms of $V^N\left(t_1, c^N(0) e^{\int_0^{t_1} \rho_s ds}\right)$. Since it is suboptimal to do a trade

$$\tilde{\xi} := \frac{y - c^N(0)}{e^{-\int_0^{t_1} \rho_s ds} + K_{t_1} c^N(0)} > 0,$$

we get

$$V^N\left(t_1, y e^{\int_0^{t_1} \rho_s ds}\right) \leq \left(1 + \frac{K_{t_1}}{2} \tilde{\xi}\right) \tilde{\xi} + \left(1 + K_{t_1} \tilde{\xi}\right)^2 V^N\left(t_1, \frac{y e^{\int_0^{t_1} \rho_s ds} - \tilde{\xi}}{1 + K_{t_1} \tilde{\xi}}\right).$$

This is exactly what we used in the last inequality of (2.22) since

$$\frac{y e^{\int_0^{t_1} \rho_s ds} - \tilde{\xi}}{1 + K_{t_1} \tilde{\xi}} = c^N(0) e^{\int_0^{t_1} \rho_s ds}.$$

Rearranging (2.22) yields

$$0 \leq a_N + b_N V^N\left(t_1, c^N(0) e^{\int_0^{t_1} \rho_s ds}\right) e^{-2 \int_0^{t_1} \rho_s ds} - \epsilon \frac{(1 + K_0 y)^2}{2K_0} \tag{2.23}$$

with

$$\begin{aligned}
a_N &:= \left(1 + \frac{K_{t_1}}{2} \frac{y - c^N(0)}{e^{-\int_0^{t_1} \rho_s ds} + K_{t_1} c^N(0)}\right) \frac{y - c^N(0)}{e^{-\int_0^{t_1} \rho_s ds} + K_{t_1} c^N(0)} e^{-2 \int_0^{t_1} \rho_s ds} \\
&\quad - \left(1 + \frac{K_0}{2} \frac{y - c^N(0)}{1 + K_0 c^N(0)}\right) \frac{y - c^N(0)}{1 + K_0 c^N(0)}, \\
b_N &:= \left(1 + K_{t_1} \frac{y - c^N(0)}{e^{-\int_0^{t_1} \rho_s ds} + K_{t_1} c^N(0)}\right)^2 - \left(1 + K_0 \frac{y - c^N(0)}{1 + K_0 c^N(0)}\right)^2.
\end{aligned}$$

Because of the continuity of K , $|a_N|$ and $|b_N|$ become arbitrarily small the higher we choose \tilde{N} . Moreover, it is suboptimal to trade the entire order at once and $c^N(0) \leq \bar{c}(0)$ such that

$$0 \leq V^N \left(t_1, c^N(0) e^{\int_0^{t_1} \rho_s ds} \right) \leq \left(1 + \frac{K_{t_1}}{2} \bar{c}(0) \right) \bar{c}(0) e^{2 \int_0^{t_1} \rho_s ds} \xrightarrow{N \rightarrow \infty} \left(1 + \frac{K_0}{2} \bar{c}(0) \right) \bar{c}(0).$$

Therefore inequality (2.23) leads to the desired contradiction. \square

Remark 2.2.19. (Limit of the barrier in the number of trading instances).

It is intuitively clear that trading gets less urgent the more trading instances are available. Therefore $j \mapsto c^{N_j}(0)$ should be increasing. But when one looks at the discrete time induction, it turns out that the proof of this monotonicity is not at all obvious. It would in particular guarantee the existence of the limit $\hat{c}(0) := \lim_{j \rightarrow \infty} c^{N_j}(0) \in (0, \infty]$. However, we could only say that $c(0) < \hat{c}(0)$. That is we have so far not excluded that the functions $L^N(0, \cdot)$ could get arbitrarily flat on $(c(0), \hat{c}(0))$ for large N .

Consider $N = 2$ and 3 as well as $(K_s)_{s \in [0,1]}$ with

$$K_0 = K_{\frac{1}{2}} = K_1 \ll K_{\frac{1}{3}} = K_{\frac{2}{3}}.$$

Then the barrier must not be increasing in N . Therefore, it is reasonable to look at the subsequence $N_j = 2^j$ such that the sets of trading instances satisfy

$$\left\{ n \frac{T}{N_j} \mid n = 0, \dots, N_j \right\} \subset \left\{ n \frac{T}{N_{j+1}} \mid n = 0, \dots, N_{j+1} \right\}.$$

2.2.4 Closed form wait and buy region

We have seen in the previous section, that for deterministic, continuous dynamics of K , an optimal strategy in continuous time exists and the value function has WR-BR structure. Let us now calculate optimal strategies and their corresponding barriers explicitly. Similar to Bank and Becherer (2009) and as explained in Gregory and Lin (1996), we use the Euler-Lagrange formalism, at least to find a candidate optimal strategy heuristically. Therefore, it is sensible to work under *Assumption Smooth*. The verification of optimality is then done by direct calculation. Let us state the main result of this subsection. Without loss of generality, we set $t = 0$.

Theorem 2.2.20. (Euler-Lagrange optimal barrier).

Suppose that Assumption Smooth holds and put

$$\tilde{t} := \inf \{ t \in [0, T] \mid K'_t + \rho_t K_t \geq 0 \}, \quad f_t := \frac{K'_t + \rho_t K_t}{K'_t + 2\rho_t K_t}.$$

Furthermore, assume that $f'_t + \rho_t f_t \geq 0$ on $[\tilde{t}, T]$. Then

$$c(t) = \left\{ \begin{array}{ll} \infty & \text{if } t \in [0, \tilde{t}] \\ \frac{1}{f_t} \left(\int_t^T \frac{f'_s + \rho_s f_s}{K_s} ds + \frac{1 - f_T}{K_T} \right) & \text{otherwise} \end{array} \right\}.$$

In particular, the optimal strategy $\Theta \in \mathcal{A}_0(x)$ for $\delta = 0$ satisfies $\frac{x - \Theta_t}{D_t} = c(t)$ on $(0, T]$.

Remark 2.2.21. The assumption $f'_t + \rho_t f_t \geq 0$ on $[\tilde{t}, T]$ is technical and does not have an economic interpretation. It is, e.g., satisfied in case of constant resilience and convex, increasing K .

Theorem 2.2.20, containing the optimal barrier c , will be a direct consequence of the following proposition, where we state the optimal strategy Θ .

Proposition 2.2.22. (Euler-Lagrange optimal strategy).

Under the assumptions of Theorem 2.2.20 and

$$\delta^* := \frac{x + \frac{\delta e^{-\int_0^{\tilde{t}} \rho_t dt}}{K_{\tilde{t}}}}{\int_{\tilde{t}}^T \frac{f'_t + \rho_t f_t}{K_t} dt + \frac{f_{\tilde{t}}}{K_{\tilde{t}}} + \frac{1-f_T}{K_T}},$$

we get for $\delta e^{-\int_0^{\tilde{t}} \rho_t dt} \leq \delta^ f_{\tilde{t}}$ that*

$$U(0, \delta, x) = (\delta^*)^2 \left(\int_{\tilde{t}}^T f_t \frac{f'_t + \rho_t f_t}{K_t} dt + \frac{f_{\tilde{t}}^2}{2K_{\tilde{t}}} + \frac{1-f_T^2}{2K_T} \right) - \frac{\left(\delta e^{-\int_0^{\tilde{t}} \rho_t dt} \right)^2}{2K_{\tilde{t}}}.$$

The corresponding unique optimal strategy $\Theta \in \mathcal{A}_0(x)$ can be expressed as

$$\Theta_t = \frac{1}{K_{\tilde{t}}} \left(\delta^* f_{\tilde{t}} - \delta e^{-\int_0^{\tilde{t}} \rho_s ds} \right) \mathbb{I}_{(\tilde{t}, \infty)}(t) + \delta^* \int_{\tilde{t} \wedge t}^t \frac{1}{K_s} (f'_s + \rho_s f_s) ds. \quad (2.24)$$

The given optimal strategy is constant on $[0, \tilde{t}]$ and afterwards

$$\begin{aligned} \Delta \Theta_{\tilde{t}} &= \delta^* \frac{f_{\tilde{t}}}{K_{\tilde{t}}} - \frac{\delta e^{-\int_0^{\tilde{t}} \rho_t dt}}{K_{\tilde{t}}}, \\ d\Theta_t &= \delta^* \frac{f'_t + \rho_t f_t}{K_t} dt, \\ \Delta \Theta_T &= \delta^* \frac{1-f_T}{K_T}. \end{aligned}$$

For $\delta = 0$, the strategy is linear in the total position x . The proof of Proposition 2.2.22 consists of several lemmata. The first one justifies that we can without loss of generality set $\tilde{t} = 0$ in the remaining proof of Proposition 2.2.22.

Lemma 2.2.23. (Wait if decrease of K outweighs resilience).

If Assumption Smooth holds and $K'_t + \rho_t K_t < 0$ for some $t \in [0, T)$, then $Br_t = \emptyset$.

Proof. The following calculation shows $K_{t+\epsilon} < K_t e^{-\rho_t \epsilon}$ for small $\epsilon > 0$:

$$K'_t = \lim_{\epsilon \searrow 0} \frac{K_{t+\epsilon} - K_t}{\epsilon} < -\rho_t K_t = \lim_{\epsilon \searrow 0} \frac{K_t e^{-\rho_t \epsilon} - K_t}{\epsilon}.$$

Therefore one gets lower costs and lower $D_{t+\epsilon}$ by postponing the trade to time $t + \epsilon$, i.e., it cannot be optimal to trade at t . \square

In the remaining proof of the proposition, we are going to exploit the fact that there is a one-to-one correspondence between Θ and D . The idea is to rewrite the cost term, which is essentially $\int_0^T D_t d\Theta_t$, in terms of the deviation process D by applying

$$d\Theta_t = \frac{dD_t + \rho_t D_t dt}{K_t}. \quad (2.25)$$

We can then use the Euler-Lagrange equation to find necessary conditions on the optimal D . Under our assumptions, these conditions turn out to be sufficient and the optimal D directly gives us an optimal Θ . Motivated by Corollary 2.2.14, we concentrate on the set of strategies $\mathcal{A}_0^c(x) \subset \mathcal{A}_0(x)$ with impulse trades at $t = 0$ and $t = T$ only.

Nevertheless, it is not clear how to account for the constraint of increasing strategies in the Euler-Lagrange equation. Therefore, we temporarily allow both buy and artificial sell trades in Lemma 2.2.24. This gives rise to the definition

$$\mathcal{A}_{0,\uparrow}^c(x) := \left\{ \Theta : [0, T+] \rightarrow \mathbb{R} \text{ with bounded variation,} \right. \\ \left. \text{continuous on } (0, T], \Theta_0 = 0, \Theta_{T+} = x \right\} \supset \mathcal{A}_0^c(x).$$

As we will see in Lemma 2.2.25, the assumptions of Proposition 2.2.22 guarantee that this procedure returns an optimal strategy consisting only of buy trades. After stating Lemma 2.2.24 and 2.2.25, the proof of Proposition 2.2.22 will be straightforward.

Lemma 2.2.24. (Optimal possibly non-monotonic strategy).

If Assumption Smooth holds and $K_t' + 2\rho_t K_t > 0$ on $[0, T]$, then

$$\Delta\Theta_0 = \delta^* \frac{f_0}{K_0} - \frac{\delta}{K_0}, \quad d\Theta_t = \delta^* \frac{f_t' + \rho_t f_t}{K_t} dt, \quad \Delta\Theta_T = \delta^* \frac{1 - f_T}{K_T}$$

is the unique optimal strategy to the auxiliary problem

$$\inf_{\Theta \in \mathcal{A}_{0,\uparrow}^c(x)} J(0, \delta, \Theta).$$

Proof. According to (2.25), the deviation process given by $D_0 = \delta$, $D_t = \delta^* f_t$ on $(0, T]$ and $D_{T+} = \delta^*$ belongs to the conjectured optimal strategy Θ from (2.24). In terms of D , the corresponding trading costs are

$$\begin{aligned} J(0, \delta, \Theta) &= \int_{(0,T)} D_t d\Theta_t + \left(\delta + \frac{K_0}{2} \Delta\Theta_0 \right) \Delta\Theta_0 + \left(D_T + \frac{K_T}{2} \Delta\Theta_T \right) \Delta\Theta_T \\ &= \int_{(0,T)} \frac{D_t}{K_t} dD_t + \int_{(0,T)} \frac{\rho_t D_t^2}{K_t} dt + \frac{D_{0+}^2 - D_0^2}{2K_0} + \frac{D_{T+}^2 - D_T^2}{2K_T}. \end{aligned}$$

Let us now look at alternative strategies $\hat{\Theta} \in \mathcal{A}_{0,\uparrow}^c(x)$ with corresponding $\hat{D} = D + h$ and show that these alternative strategies cause higher trading costs than Θ . That is in the following, we work with functions $h : [0, T+] \rightarrow \mathbb{R}$ which are of bounded variation

and continuous on $(0, T]$ with $h_0 = 0$ and possible jumps $(h_{0+} - h_0), (h_{T+} - h_T) \in \mathbb{R}$. Using

$$\Delta \hat{\Theta}_0 = \Delta \Theta_0 + \frac{h_{0+}}{K_0}, \quad d\hat{\Theta}_t = d\Theta_t + \frac{dh_t + \rho_t h_t dt}{K_t}, \quad \Delta \hat{\Theta}_T = \Delta \Theta_T + \frac{h_{T+} - h_T}{K_T},$$

a straightforward calculation yields

$$\begin{aligned} J(0, \delta, \hat{\Theta}) &= \int_{(0,T)} \hat{D}_t d\hat{\Theta}_t + \left(\delta + \frac{K_0}{2} \Delta \hat{\Theta}_0 \right) \Delta \hat{\Theta}_0 + \left(\hat{D}_T + \frac{K_T}{2} \Delta \hat{\Theta}_T \right) \Delta \hat{\Theta}_T \\ &= J(0, \delta, \Theta) + \Delta J_1 + \Delta J_2, \end{aligned}$$

$$\begin{aligned} \Delta J_1 &:= \int_{(0,T)} \frac{2\rho_t D_t h_t}{K_t} dt + \int_{(0,T)} \frac{h_t}{K_t} dD_t + \int_{(0,T)} \frac{D_t}{K_t} dh_t \\ &\quad + \frac{D_{0+} h_{0+}}{K_0} + \frac{D_{T+} h_{T+} - D_T h_T}{K_T}, \\ \Delta J_2 &:= \int_{(0,T)} \frac{\rho_t h_t^2}{K_t} dt + \int_{(0,T)} \frac{h_t}{K_t} dh_t + \frac{h_{0+}^2}{2K_0} + \frac{h_{T+}^2 - h_T^2}{2K_T}. \end{aligned}$$

Notice that we collect all terms containing D in ΔJ_1 . We are now going to finish the proof by showing that $\Delta J_1 = 0$ and $\Delta J_2 > 0$ if h does not vanish.

Let us first rewrite ΔJ_1 exploiting the fact that $D_t = \delta^* f_t$, use integration by parts, the definition of f and again integration by parts to get

$$\begin{aligned} &\Delta J_1 \\ &= \delta^* \left\{ \int_{(0,T)} \frac{2\rho_t f_t h_t}{K_t} dt + \int_{(0,T)} \frac{h_t}{K_t} df_t + \int_{(0,T)} \frac{f_t}{K_t} dh_t + \frac{f_0 h_{0+}}{K_0} + \frac{h_{T+} - f_T h_T}{K_T} \right\} \\ &= \delta^* \left\{ \int_{(0,T)} \frac{2\rho_t K_t + K_t'}{K_t^2} f_t h_t dt + \frac{h_{T+}}{K_T} \right\} \\ &= \delta^* \left\{ \int_{(0,T)} \frac{\rho_t h_t}{K_t} dt + \frac{h_{T+}}{K_T} + \int_{(0,T)} \frac{K_t'}{K_t^2} h_t dt \right\} \\ &= \delta^* \left\{ \int_{(0,T)} \frac{\rho_t h_t}{K_t} dt + \int_{(0,T)} \frac{1}{K_t} dh_t + \frac{h_{0+}}{K_0} + \frac{h_{T+} - h_T}{K_T} \right\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} x &= \int_{(0,T)} d\hat{\Theta}_t + \Delta \hat{\Theta}_0 + \Delta \hat{\Theta}_T \\ &= \left(\int_{(0,T)} d\Theta_t + \Delta \Theta_0 + \Delta \Theta_T \right) \\ &\quad + \left(\int_{(0,T)} \frac{\rho_t h_t}{K_t} dt + \int_{(0,T)} \frac{1}{K_t} dh_t + \frac{h_{0+}}{K_0} + \frac{h_{T+} - h_T}{K_T} \right) \\ &= x + \frac{\Delta J_1}{\delta^*}. \end{aligned}$$

That is $\Delta J_1 = 0$. Hence, $J(\hat{\Theta}) - J(\Theta) = \Delta J_2$. Applying integration by parts on the dh_t integral yields

$$\Delta J_2 = \int_{(0,T)} \frac{h_t^2}{2K_t} \left(2\rho_t + \frac{K'_t}{K_t} \right) dt + \frac{h_{T+}^2}{2K_T}.$$

Due to our assumption $2\rho_t + \frac{K'_t}{K_t} > 0$ on $[0, T]$, ΔJ_2 is positive as desired. \square

Lemma 2.2.25. (Exclude artificial selling).

If Assumption Smooth holds, $\tilde{t} = 0$ and $f'_t + \rho_t f_t \geq 0$ on $[0, T]$, then $K'_t + \rho_t K_t \geq 0$ and $f_t \in [0, 1)$ on this interval.

Proof. The function $t \mapsto K'_t + \rho_t K_t$ is differentiable and non-negative at $t = 0$. Assume it hits zero at $\check{s} \in [0, T)$ and suppose for a contradiction that there exists $\hat{s} \in (\check{s}, T)$ such that $K'_s + \rho_s K_s < 0$. If necessary, modify \hat{s} such that $K'_s + 2\rho_s K_s > 0$. Then $f_s = 0$ and $f'_s < 0$. Therefore, there exists $\tilde{s} \in (\check{s}, \hat{s}]$ such that $f_s < 0$ and $f'_s < 0$. That is $f'_s + \rho_s f_s < 0$, which is a contradiction. \square

Thanks to Lemma 2.2.25, we know that the strategy (2.24) does not contain any negative trades. It also shows that Proposition 2.2.22 only deals with dynamics of K that result in a continuous function f , since $K'_t + 2\rho_t K_t > 0$. In particular, the cost function J is strictly convex in the strategy as we see from (2.15), i.e. uniqueness is guaranteed.

Proof of Proposition 2.2.22. Thanks to Lemma 2.2.23, we can without loss of generality set $\tilde{t} = 0$. We have $\mathcal{A}_0(x) \supset \mathcal{A}_0^c(x) \subset \mathcal{A}_{0,\uparrow}^c(x)$ and Corollary 2.2.14 guarantees

$$\inf_{\Theta \in \mathcal{A}_0(x)} J(0, \delta, \Theta) = \inf_{\Theta \in \mathcal{A}_0^c(x)} J(0, \delta, \Theta) \geq \inf_{\Theta \in \mathcal{A}_{0,\uparrow}^c(x)} J(0, \delta, \Theta).$$

Due to Lemma 2.2.24, which we can apply thanks to Lemma 2.2.25, we know that (2.24) is the unique optimal strategy within the set $\mathcal{A}_{0,\uparrow}^c(x)$. Lemma 2.2.25 assures also that (2.24) does not contain any negative trades, i.e. belongs to $\mathcal{A}_0^c(x)$. Therefore, it must also be the unique optimal strategy within this smaller strategy set and is also optimal within $\mathcal{A}_0(x)$. \square

Proof of Theorem 2.2.20. For deterministic, continuous K , we proved in Theorem 2.2.15 that the value function has WR-BR structure. Therefore the unique optimal strategy from Proposition 2.2.22 can be used to calculate the corresponding barrier. Imagine that $\tilde{t} = 0$ and we want to trade x shares on $[0, T]$ with $D_0 = 0$. From (2.24), we then get

$$c(0) = \frac{x - \Delta\Theta_0}{K_0 \Delta\Theta_0} = \frac{x}{\delta^* f_0} - \frac{1}{K_0} = \frac{1}{f_0} \left(\int_0^T \frac{f'_s + \rho_s f_s}{K_s} ds + \frac{1 - f_T}{K_T} \right).$$

Without loss of generality, we have formulated Proposition 2.2.22 for initial trading time $t = 0$. It also holds for $t \in [0, T]$. This yields the desired barrier. \square

Let us now illustrate our results by discussing some examples. For simplicity, take constant resilience $\rho > 0$. Then the condition $f'_t + \rho f_t \geq 0$ is equivalent to

$$2\rho^2 K_t + 3\rho K'_t + K''_t \geq 0. \quad (2.26)$$

Example 2.2.26. (Constant price impact $K_t \equiv \kappa$).

We recover our results from Subsection 2.2.1, $f(t) \equiv \frac{1}{2}$ and $\delta^* = 2\frac{\kappa x + \delta}{\rho T + 2}$.

Example 2.2.27. (Exponential price impact $K_t = \kappa e^{\nu \rho t}$).

Let $\nu \in \mathbb{R}$ be the slope of the exponential price impact relative to the resilience. Condition (2.26) is satisfied for $\nu \geq -1$. Due to $f(t) \equiv \frac{\nu+1}{\nu+2}$, we have constant D_t just as in the constant impact case. The optimal strategy $\Theta \in \mathcal{A}_0(x)$ for $\delta = 0$ is

$$\Delta\Theta_0 = \frac{x\nu(\nu+1)}{1 - e^{-\nu\rho T} + \nu(\nu+2)}, \quad d\Theta_t = \rho e^{-\nu\rho t} \Delta\Theta_0 dt, \quad \Delta\Theta_T = \frac{x\nu e^{-\nu\rho T}}{1 - e^{-\nu\rho T} + \nu(\nu+2)}.$$

The initial trade $\Delta\Theta_0$ clearly approaches x for $\nu \nearrow \infty$ and the entire strategy does not depend on κ . The corresponding barrier

$$\tilde{c}(t) = \frac{K_t}{\delta^* f_0} \left(x - \Delta\Theta_0 - \int_0^t d\Theta_s \right) = \frac{1 + \nu - e^{-\rho\nu(T-t)}}{\nu(1 + \nu)}$$

is decreasing in time and ν . That is for small ν , the wait region is large since it is attractive to trade close to maturity.

Instead of using the Euler-Lagrange formalism, this example could also be discussed as an unconstrained optimization in discrete time. That is one would calculate the optimal strategy by backward induction ignoring the constraint that only positive trades are admissible. This gives an optimal barrier between a buy and an artificial sell region. For $\nu \geq -1$, it coincides with the sought-after buy and wait region barrier since only upward reflections from the barrier occur.

For $\nu < -1$, $K_t e^{-\rho(T-t)} > K_T$ for all $t \in [0, T)$. Hence, it is optimal to trade the entire order at T . In this situation, $c(t) \equiv \infty$ since the decrease of K_t outweighs the resilience.

Example 2.2.28. (Straight-line price impact $K_t = \kappa + mt$).

Let $m > -\frac{\kappa}{T}$ such that K is positive. Condition (2.26) is satisfied for $m \geq -\frac{2\kappa\rho}{3+2\rho T}$. One can explicitly state the optimal strategy and barrier function in this case. As in the previous example, we can check that it is optimal for

$$m \in \left(-\frac{\kappa}{T}, -\frac{\kappa}{T} (1 - e^{-\rho T}) \right]$$

to trade the entire order at T . However, the presented methods do not work for

$$m \in \left(-\frac{\kappa}{T} (1 - e^{-\rho T}), -\frac{2\kappa\rho}{3 + 2\rho T} \right).$$

2.3 Geometric Brownian motion price impact, discrete time

In the last section, we have seen how one can generalize the work of Obizhaeva and Wang (2006) by allowing for deterministic price impact dynamics. This allows the large investor to refine the impact model by fitting it, e.g., to drifts or trends that are present in historic data. However, the investor deterministically fixes $(K_s)_{s \in [t, T]}$ at the beginning of the trading period and so the optimal strategy is also deterministically fixed at initial time t . There are no adjustments due to new information that may become available during the trading period. But this is exactly what one would like to have, since it might, e.g., be sensible to accelerate trading when there happens to be a lot of liquidity available in the order book. We get this feature of differential order placement as soon as we include a diffusive term in our price impact model.

In this section, let us consider the time-inhomogeneous *geometric Brownian motion* (GBM) as our illiquidity process. According to Lemma 1.3.1 and 1.3.3, the linear structure of the GBM simplifies the optimization to a two-dimensional problem. But even for this simplified problem, we cannot find a closed form solution. But under *Assumption SpecialGBM*, which, e.g., covers the time-homogeneous GBM, we can show in our main result Proposition 2.3.1 that the discrete time value function has WR-BR structure. This does not hold for all time-inhomogeneous GBM as we will see in Subsection 2.5.3.

Proposition 2.3.1. (SpecialGBM, discrete time case: WR-BR structure).

Under Assumption SpecialGBM, the discrete time value function has WR-BR structure.

a) *If $\int_{t_n}^T (\bar{\mu}_s + \rho_s) ds \leq 0$ for all $n = 0, \dots, N - 1$, then $c^N(t_n, \kappa) \equiv \infty$ and*

$$U^N(t_n, \delta, x, \kappa) = \left(\delta e^{-\int_{t_n}^T \rho_s ds} + \frac{\kappa}{2} e^{\int_{t_n}^T \bar{\mu}_s ds} x \right). \quad (2.27)$$

b) *If $\int_{t_n}^{t_{n+1}} (\bar{\mu}_s + \rho_s) ds > 0$ for all $n = 0, \dots, N - 1$, then $c^N(t_n, \kappa) = \frac{\tilde{c}_n}{\kappa}$ for $\tilde{c}_n \in (0, \infty)$ and $W^N(t_n, \cdot)$ is continuously differentiable at \tilde{c}_n , three times continuously differentiable everywhere else and $\mathcal{B}^W(W^N)(t_n, \cdot)$ is convex and decreasing.*

Recall the notation W^N and \mathcal{B}^W from (1.23) and (1.26). In order to show Proposition 2.3.1 by backward induction, we need two lemmata, whose proofs are postponed to the end of this subsection. The first lemma is independent of the choice of K and does not use the existence of an optimal strategy.

Lemma 2.3.2. (Wait region adjusted dynamic programming).

Let $U^N(t_{n+1}, \cdot, \cdot, \cdot)$ be continuous in δ and x , and consider (t_n, δ, x, κ) such that $\frac{x}{\delta} \in WR_{t_n}^N(\kappa)$. Recall from (2.6) that $a_n := \exp(-\int_{t_n}^{t_{n+1}} \rho_s ds)$. Then

$$U^N(t_n, \delta, x, \kappa) = \mathbb{E}_{t_n, \delta, \kappa} [U^N(t_{n+1}, \delta a_n, x, K_{t_{n+1}})]. \quad (2.28)$$

Lemma 2.3.3. *Let $\tilde{c} \in (0, \infty)$, φ a density function on \mathbb{R} and $h : [0, \infty) \rightarrow [0, \infty)$ continuously differentiable at \tilde{c} , three times continuously differentiable everywhere else and $\mathcal{B}^W(h)$ convex. Then $\mathcal{B}^W(\tilde{h})$ is convex, where*

$$\tilde{h}(z) := \int_{-\infty}^{\infty} e^{-r} h(ze^r) \varphi(r) dr.$$

Proof of Proposition 2.3.1. a) Since W^N corresponding to (2.27) satisfies $\mathcal{B}^W(W^N) > 0$ and $\mathcal{W}^W(W^N) = 0$ everywhere, it is optimal to trade everything at maturity.

b) Omit N in W^N for notational convenience. The proof is by backward induction with induction step from t_{n+1} to t_n . Rewrite (2.28) from Lemma 2.3.2 as

$$\begin{aligned} W(t_n, z) &= \mathbb{E}_{t_n, \delta, \kappa} \left[a_n^2 \frac{\kappa}{K_{t_{n+1}}} W \left(t_{n+1}, z \left(a_n^{-1} \frac{K_{t_{n+1}}}{\kappa} \right) \right) \right] \\ &= a_n \int_{-\infty}^{\infty} e^{-r} W(t_{n+1}, ze^r) \varphi_n(r) dr =: W^{WR}(t_n, z). \end{aligned}$$

We used that $\ln \left(a_n^{-1} \frac{K_{t_{n+1}}}{\kappa} \right)$ has the following Gaussian density under the measure $\mathbb{P}_{t_n, \kappa}$:

$$\varphi_n(r) := \left(2\pi \int_{t_n}^{t_{n+1}} \bar{\sigma}_s^2 ds \right)^{-\frac{1}{2}} \exp \left(-\frac{\left(r - \int_{t_n}^{t_{n+1}} \rho_s + \bar{\mu}_s - \frac{1}{2} \bar{\sigma}_s^2 ds \right)^2}{2 \int_{t_n}^{t_{n+1}} \bar{\sigma}_s^2 ds} \right).$$

Since $W(t_{n+1}, \cdot)$ is continuously differentiable, we can define the smallest z such that the wait region adjoins to the buy region

$$\tilde{c}_n := \inf \{ z \in [0, \infty) \mid \mathcal{B}^W(W^{WR})(t_n, z) \leq 0 \}, \quad \inf \emptyset := \infty.$$

For $\tilde{c}_n < \infty$, set

$$W^{BR}(t_n, z) := \frac{W^{WR}(t_n, \tilde{c}_n) + \frac{1}{2}(z + 1)^2}{(1 + \tilde{c}_n)^2} - \frac{1}{2}.$$

Then $\mathcal{B}^W(W^{BR})(t_n, z) \equiv 0$ and $W^{WR}(t_n, \tilde{c}_n) = W^{BR}(t_n, \tilde{c}_n)$. Our aim is to prove

$$W(t_n, z) = \begin{cases} W^{BR}(t_n, z) & \text{if } z \in [\tilde{c}_n, \infty) \\ W^{WR}(t_n, z) & \text{otherwise} \end{cases}. \quad (2.29)$$

This can be done by showing that $\mathcal{B}^W(W^{WR})(t_n, \cdot)$ is decreasing. As an induction

hypothesis, we use that $\mathcal{B}^W(W)(t_{n+1}, \cdot)$ is decreasing.

$$\begin{aligned}
& \frac{\partial}{\partial z} \mathcal{B}^W(W^{WR})(t_n, z) = \partial_z W^{WR}(t_n, z) - (1+z) \partial_{zz} W^{WR}(t_n, z) \\
&= a_n \int_{-\infty}^{\infty} [\partial_z W(t_{n+1}, ze^r) - (1+z)e^r \partial_{zz} W(t_{n+1}, ze^r)] \varphi_n(r) dr \\
&= a_n \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial z} \mathcal{B}^W(W)(t_{n+1}, ze^r) + (1-e^r) \partial_{zz} W(t_{n+1}, ze^r) \right] \varphi_n(r) dr \\
&\leq a_n \int_{-\infty}^{\infty} (1-e^r) \partial_{zz} W(t_{n+1}, ze^r) \varphi_n(r) dr
\end{aligned}$$

Exploiting $\tilde{c}_{n+1} \in (0, \infty)$ and $W(t_{n+1}, z) = W^{BR}(t_{n+1}, z)$ for $z \geq \tilde{c}_{n+1}$ yields

$$\lim_{z \rightarrow \infty} \frac{\partial}{\partial z} \mathcal{B}^W(W^{WR})(t_n, z) \leq 2a_n \frac{W^{WR}(t_{n+1}, \tilde{c}_{n+1}) + \frac{1}{2}}{(1 + \tilde{c}_{n+1})^2} \left(1 - e^{\int_{t_n}^{t_{n+1}} (\bar{\mu}_s + \rho_s) ds} \right).$$

That is $\mathcal{B}^W(W^{WR})(t_n, \cdot)$ is decreasing for large z and convex thanks to Lemma 2.3.3. Therefore, it must be decreasing everywhere. This proves (2.29).

Due to Lemma 2.1.2 and $\lim_{z \rightarrow \infty} \frac{\partial}{\partial z} \mathcal{B}^W(W^{WR})(t_n, z) < 0$, $\tilde{c}_n \in (0, \infty)$. It is clear that $W(t_n, \cdot)$ is three times continuously differentiable on $[0, \infty) \setminus \{\tilde{c}_n\}$. We have

$$\mathcal{B}^W(W^{WR})(t_n, \tilde{c}_n) = \mathcal{B}^W(W^{BR})(t_n, \tilde{c}_n) = 0,$$

which is equivalent to $\partial_z W^{WR}(t_n, \tilde{c}_n) = \partial_z W^{BR}(t_n, \tilde{c}_n)$. That is $W(t_n, \cdot)$ is continuously differentiable at \tilde{c}_n . \square

Proof of Lemma 2.3.2. Due to the dynamic programming principle, we can write

$$\begin{aligned}
U^N(t_n, \delta, x, \kappa) &= \inf_{\xi \in [0, x]} \left\{ \left(\delta + \frac{\kappa}{2} \xi \right) \xi + \mathbb{E}_{t_n, \delta, \kappa} [U^N(t_{n+1}, (\delta + \kappa \xi) a_n, x - \xi, K_{t_{n+1}})] \right\} \\
&\leq \mathbb{E}_{t_n, \delta, \kappa} [U^N(t_{n+1}, \delta a_n, x, K_{t_{n+1}})].
\end{aligned} \tag{2.30}$$

Suppose for a contradiction that

$$U^N(t_n, \delta, x, \kappa) < \mathbb{E}_{t_n, \delta, \kappa} [U^N(t_{n+1}, \delta a_n, x, K_{t_{n+1}})].$$

Hence, the assumed continuity guarantees the existence of $\xi \in (0, x)$ such that

$$U^N(t_n, \delta, x, \kappa) = \left(\delta + \frac{\kappa}{2} \xi \right) \xi + \mathbb{E}_{t_n, \delta, \kappa} [U^N(t_{n+1}, (\delta + \kappa \xi) a_n, x - \xi, K_{t_{n+1}})]. \tag{2.31}$$

We get a contradiction to $\frac{x}{\delta} \in WR_{t_n}^N(\kappa)$ by applying (2.30) in (2.31), i.e.,

$$U^N(t_n, \delta, x, \kappa) \geq \left(\delta + \frac{\kappa}{2} \xi \right) \xi + U^N(t_n, \delta + \kappa \xi, x - \xi, \kappa).$$

\square

Proof of Lemma 2.3.3. Compute

$$\begin{aligned}\frac{\partial}{\partial z}\mathcal{B}^W(\tilde{h}) &= \tilde{h}_z - (1+z)\tilde{h}_{zz}, & \frac{\partial^2}{\partial z^2}\mathcal{B}^W(\tilde{h}) &= -(1+z)\tilde{h}_{zzz}, \\ \tilde{h}_z(z) &= \int_{-\infty}^{\infty} h_z(ze^r)\varphi(r)dr, & \tilde{h}_{zz}(z) &= \int_{-\infty}^{\infty} e^r h_{zz}(ze^r)\varphi(r)dr.\end{aligned}$$

Then $u \mapsto \int_{-\infty}^u e^r h_{zz}(ze^r)\varphi(r)dr$ is continuous at $u = \ln\left(\frac{\tilde{c}}{z}\right)$. But

$$u \mapsto \frac{\partial}{\partial z} \int_{-\infty}^u e^r h_{zz}(ze^r)\varphi(r)dr = \int_{-\infty}^u e^{2r} h_{zzz}(ze^r)\varphi(r)dr$$

has a jump at $u = \ln\left(\frac{\tilde{c}}{z}\right)$ of size

$$\begin{aligned}& \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial z} \int_{\ln\left(\frac{\tilde{c}}{z}\right) - \frac{\epsilon}{2}}^{\ln\left(\frac{\tilde{c}}{z}\right) + \frac{\epsilon}{2}} e^r h_{zz}(ze^r)\varphi(r)dr \\ &= \lim_{\epsilon \rightarrow 0} \int_{\ln\left(\frac{\tilde{c}}{z}\right) - \frac{\epsilon}{2}}^{\ln\left(\frac{\tilde{c}}{z}\right) + \frac{\epsilon}{2}} e^{2r} \frac{h_{zz}(\tilde{c}+) - h_{zz}(\tilde{c}-)}{\epsilon} \varphi(r)dr \\ &= \frac{\tilde{c}^2}{z^2} [h_{zz}(\tilde{c}+) - h_{zz}(\tilde{c}-)] \varphi\left(\ln\left(\frac{\tilde{c}}{z}\right)\right).\end{aligned}$$

That is

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\mathcal{B}^W(\tilde{h})(z) &= -(1+z)\tilde{h}_{zzz}(z) \\ &= -(1+z) \left\{ \int_{-\infty}^{\infty} e^{2r} h_{zzz}(ze^r)\varphi(r)dr + \frac{\tilde{c}^2}{z^2} [h_{zz}(\tilde{c}+) - h_{zz}(\tilde{c}-)] \varphi\left(\ln\left(\frac{\tilde{c}}{z}\right)\right) \right\} \\ &= \int_{-\infty}^{\infty} e^{2r} \frac{\partial^2}{\partial z^2}\mathcal{B}^W(h)(ze^r) \frac{1+z}{1+ze^r} \varphi(r)dr \\ &\quad - (1+z) \frac{\tilde{c}^2}{z^2} [h_{zz}(\tilde{c}+) - h_{zz}(\tilde{c}-)] \varphi\left(\ln\left(\frac{\tilde{c}}{z}\right)\right).\end{aligned}$$

This term is positive since $\mathcal{B}^W(h)$ is convex and therefore $\frac{\partial}{\partial z}\mathcal{B}^W(h)(\tilde{c}+) > \frac{\partial}{\partial z}\mathcal{B}^W(h)(\tilde{c}-)$ which is equivalent to $h_{zz}(\tilde{c}+) < h_{zz}(\tilde{c}-)$. \square

Remark 2.3.4. (GBM: Situation at t_{N-1}).

We can explicitly state the value function and the barrier at t_{N-1}

$$\begin{aligned}W^{WR}(t_{N-1}, z) &= \frac{1}{2} \exp\left(\int_{t_{N-1}}^{t_N} \bar{\mu}_s ds\right) z^2 + \exp\left(-\int_{t_{N-1}}^{t_N} \rho_s ds\right) z, \\ c^N(t_{N-1}, \kappa) &= \left\{ \begin{array}{ll} \frac{1 - \exp\left(-\int_{t_{N-1}}^{t_N} \rho_s ds\right)}{\kappa \exp\left(\int_{t_{N-1}}^{t_N} \bar{\mu}_s ds\right) - \exp\left(-\int_{t_{N-1}}^{t_N} \rho_s ds\right)} & \text{if } \int_{t_{N-1}}^{t_N} (\bar{\mu}_s + \rho_s) ds > 0 \\ \infty & \text{otherwise} \end{array} \right\}.\end{aligned}$$

It is optimal not to trade at all at t_{N-1} if the negative drift of K outweighs the resilience.

Remark 2.3.5. (Time-inhomogeneous GBM: WR-BR-WR structure can occur).

At first glance, one would hope to have WR-BR structure for all time-inhomogeneous GBM. One might have guessed that a generalization of Proposition 2.3.1 a) should hold, i.e., $\int_{t_n}^{t_{n+1}} (\bar{\mu}_s + \rho_s) ds \leq 0$ leads to $\tilde{c}_n = \infty$. Astonishingly, this generalization does not apply, since trades are dependent on K . In Subsection 2.5.3, we exploit this dependence to construct a WR-BR-WR example.

In Subsection 2.4, it turns out that even for the time-homogeneous GBM, the cost function is not necessarily convex in the strategy. Therefore, it is unclear how to prove existence of an optimal strategy in continuous time in full generality and how to transfer Proposition 2.3.1 from discrete to continuous time.

Also notice that the GBM leads to $c(t, \kappa) = \frac{\tilde{c}(t)}{\kappa}$, i.e., optimal strategies are “passive in the liquidity”. This terminology is chosen in analogy to aggressive and passive in the money strategies that dynamically react to price changes; see Schied and Schöneborn (2009). Due to $z = \frac{\kappa x}{\delta}$, higher values of κ bring us from the no-trading into the trading region. This can be explained as follows. An increased κ , i.e., decreased order book height q , means according to our assumed resilience model that less fresh limit sell orders flow into the book. This leads to an incentive to do a trade and attract new limit orders, although this trade is quite expensive. The argument for decreased κ is analogous: An increase of the limit order book height leads to a decrease of the optimal extra spread. The order execution is delayed until the extra spread has sufficiently reduced, although one misses out on inexpensive κ in the meantime. Hence, there is a tradeoff between trading when κ is low and having a well balanced extra spread. In case of the GBM, the extra spread dominates the tradeoff. As we shall see later, this changes as soon as we introduce a pronounced mean-reversion in the drift of K . In the following, let us formally introduce the notion of passive and aggressive in the liquidity strategies and examine how it relates to the slope of the barrier in κ .

Aggressive versus passive in the liquidity strategies

Definition 2.3.6. (Aggressive versus passive in the liquidity).

Assume that the value function has WR-BR structure. Let $\Theta(t, \delta, x, \kappa)$ be the unique optimal strategy and $\frac{x}{\delta} > c(t, \kappa)$. Then we call the initial trade

$$\Delta\Theta_t(t, \delta, x, \kappa) = \max \left\{ 0, \frac{x - c(t, \kappa)\delta}{1 + \kappa c(t, \kappa)} \right\} \quad (2.32)$$

aggressive/passive in the liquidity if it is decreasing/increasing with respect to κ .

Thus, a barrier that increases in κ guarantees aggressive in the liquidity trades. But due to κ in the denominator of (2.32), things are not so clear in case of a decreasing barrier. Let us summarize this issue in the following proposition.

Proposition 2.3.7. (Barrier slope \leftrightarrow aggressive/passive trades).

Assume that the value function has WR-BR structure.

a) Let $c(t, \kappa)$ be increasing in κ .

Then the optimal initial trade is aggressive in the liquidity for all $\frac{x}{\delta} > c(t, \kappa)$.

b) Let $\frac{x}{\delta} > c(t, \kappa)$ and let the optimal initial trade be passive in the liquidity.

Then $c(t, \kappa)$ is strictly decreasing in κ .

c) Let K be a GBM.

Then the optimal initial trade is passive in the liquidity for all $\frac{x}{\delta} > c(t, \kappa)$.

Proof. Statement a) and b) are immediate.

For statement c), notice that $c(t, \kappa) = \frac{\tilde{c}(t)}{\kappa}$, $c + \kappa c_\kappa = 0$ and

$$\frac{\partial}{\partial \kappa} \left\{ \frac{x - c(t, \kappa)\delta}{1 + \kappa c(t, \kappa)} \right\} = -\frac{c_\kappa \delta (1 + \kappa c) + (c + \kappa c_\kappa)(x - c\delta)}{(1 + \kappa c)^2} \geq 0.$$

□

Using the GBM, which is the standard model for positive diffusions, optimal strategies are always passive in the liquidity. But in a real-life market, where the order book height is mean-reverting, aggressive in the liquidity strategies are reasonable. The investor would want to trade when liquidity is high, since it is likely to fall again afterwards. Therefore, we do not want to limit our considerations to the GBM. Instead, let us look at more general positive diffusion models for K that might better describe the properties of a real-life market by, e.g., featuring mean-reversion.

2.4 General price impact diffusion

This section is going to contain a WR-BR result which can be applied quite flexibly to various diffusions for K . We work under *Assumption SpecialDiff*. As we will show, this ensures that our cost functional J is strictly convex. The convexity guarantees uniqueness of the optimal strategy provided it exists. This uniqueness in turn excludes WR-BR-WR situations since such an upper barrier between buy and wait region would correspond to a non-uniqueness of the optimal strategy in the sense that it would be optimal to wait and to do a strictly positive trade to the lower barrier between wait and buy region. Similar to the deterministic case, we first perform this argument for the discrete trading time case and then transfer it to continuous time. Again, we get along without using the HJB equation.

However, convexity is not a necessary condition for uniqueness and *Assumption SpecialDiff* is only sufficient for convexity. Therefore, there exist situations, not covered by our proposition, in which uniqueness and WR-BR structure hold. On the other hand, the WR-BR-WR examples from Subsection 2.5 indicate that it is not possible to get along without any assumptions on K .

Example 2.4.1. (Convexity for two trading instances).

Assume that trading is allowed at $t_0 = 0$ and $t_1 = T$ only. Then

$$J(0, 0, (\xi_0, \xi_1), \kappa) = \frac{\kappa}{2}\xi_0^2 + \kappa\xi_0\xi_1e^{-\int_0^T \rho_s ds} + \frac{\mathbb{E}_{t_0, \kappa}[K_T]}{2}\xi_1^2$$

is strictly convex in the strategy $\xi = (\xi_0, \xi_1)$ if and only if $\mathbb{E}_{t_0, \kappa}[K_T] - \kappa e^{-2\int_0^T \rho_s ds} > 0$. This example clarifies that the cost functional J is not necessarily convex in general.

2.4.1 Existence of a unique optimal strategy

Under *Assumption SpecialDiff*, we show in Lemma 2.4.2 that $J(\Theta)$ is strictly convex. This guarantees the uniqueness of an optimal strategy provided it exists. We can then use the convexity together with the Komlós Theorem to finally get the existence of an optimal strategy in Proposition 2.4.3. Although we consider continuous trading time in this part, the results carry over to discrete time as well.

Lemma 2.4.2. (Costs convex in the strategy).

Let Assumption SpecialDiff hold. Then $J(t, \delta, \cdot, \kappa)$ is strictly convex on \mathcal{A}_t .

Proof. Similar as in Lemma 2.2.11 for deterministic K , we demonstrate below that

$$J(\Theta) = \frac{1}{2}\mathbb{E}_{t, \delta, \kappa} \left[\frac{D_{T+}^2}{K_T} - \frac{\delta^2}{\kappa} + \int_{[t, T]} \eta_s D_s^2 ds \right] \quad (2.33)$$

with $\eta_t := \frac{2\rho_t}{K_t} + \frac{\mu(t, K_t)}{K_t^2} - \frac{\sigma^2(t, K_t)}{K_t^3}$ from *Assumption SpecialDiff i)* being strictly positive. Therefore, the right-hand side is convex in the process $(D_s)_{s \in [t, T]}$. Moreover, D is linear in the strategy Θ . Thus, for two strategies $\Theta', \Theta'' \in \mathcal{A}_t$ with corresponding D', D'' both starting in $D'_t = D''_t = \delta$, we have $J(\nu\Theta' + (1-\nu)\Theta'') < \nu J(\Theta') + (1-\nu)J(\Theta'')$ for all $\nu \in (0, 1)$ as desired. Hence, we only need to show (2.33).

Define the local martingale $M_s := \int_{[t, s \wedge T]} \frac{D_u^2 \sigma(u, K_u)}{2K_u^2} dW_u^K$ for $s \in [t, \infty)$. That is $\tau_n = \{s \geq t \mid \langle M \rangle_s \geq n\}$ is an increasing sequence of stopping times such that $\tau_n \nearrow \infty$ a.s. and M^{τ_n} is a martingale for every n . In particular, $\mathbb{E}[M_{T \wedge \tau_n}] = 0$. Due to the monotone convergence theorem and $\tau_n \geq T$ a.s. for large n ,

$$J(\Theta) = \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{[t, T \wedge \tau_n]} \left(D_s + \frac{K_s}{2} \Delta \Theta_s \right) d\Theta_s \right].$$

Using $d\Theta_s = \frac{dD_s + \rho_s D_s ds}{K_s}$ and $\Delta \Theta_s = \frac{\Delta D_s}{K_s}$, we get

$$J(\Theta) = \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{[t, T \wedge \tau_n]} \frac{D_s + \frac{1}{2} \Delta D_s}{K_s} dD_s + \int_{[t, T \wedge \tau_n]} \frac{\rho_s D_s^2}{K_s} ds + \int_{[t, T \wedge \tau_n]} \frac{\frac{1}{2} \Delta D_s \rho_s D_s}{K_s} ds \right].$$

The last integral is zero, since D has at most countably many jumps. With integration by parts for càglàd processes,

$$\int_{[t, T \wedge \tau_n]} \frac{D_s}{K_s} dD_s = \frac{D_{(T \wedge \tau_n)^+}^2}{K_{(T \wedge \tau_n)}} - \frac{\delta^2}{\kappa} - \int_{[t, T \wedge \tau_n]} D_s d\left(\frac{D}{K}\right)_s - \sum_{s \in [t, T \wedge \tau_n]} \frac{(\Delta D_s)^2}{K_s}.$$

Use $d\left(\frac{D}{K}\right)_s = \frac{1}{K_s} dD_s + D_s d\left(\frac{1}{K_s}\right)$ and rearrange terms to get

$$\int_{[t, T \wedge \tau_n]} \frac{D_s}{K_s} dD_s = \frac{1}{2} \left(\frac{D_{(T \wedge \tau_n)^+}^2}{K_{(T \wedge \tau_n)}} - \frac{\delta^2}{\kappa} - \int_{[t, T \wedge \tau_n]} D_s^2 d\left(\frac{1}{K_s}\right) - \sum_{s \in [t, T \wedge \tau_n]} \frac{(\Delta D_s)^2}{K_s} \right).$$

Applying Itô's formula

$$dq_s = d\left(\frac{1}{K_s}\right) = \left(\frac{\sigma^2(s, K_s)}{K_s^3} - \frac{\mu(s, K_s)}{K_s^2} \right) ds - \frac{\sigma(s, K_s)}{K_s^2} dW_s^K$$

yields

$$\int_{[t, T \wedge \tau_n]} \left(D_s + \frac{K_s}{2} \Delta \Theta_s \right) d\Theta_s = \frac{1}{2} \left[\frac{D_{(T \wedge \tau_n)^+}^2}{K_{T \wedge \tau_n}} - \frac{\delta^2}{\kappa} + \int_{[t, T \wedge \tau_n]} \eta_s D_s^2 ds + M_{T \wedge \tau_n} \right].$$

The assertion follows, since *Assumption SpecialDiff* ii) and iii) together with Lebesgue's dominated convergence theorem guarantee

$$\mathbb{E} \left[\frac{D_{(T \wedge \tau_n)^+}^2}{K_{T \wedge \tau_n}} \right] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[\frac{D_{T^+}^2}{K_T} \right], \quad \mathbb{E} \left[\int_{[t, T \wedge \tau_n]} \eta_s D_s^2 ds \right] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[\int_{[t, T]} \eta_s D_s^2 ds \right].$$

□

Proposition 2.4.3. (Existence of a unique optimal strategy).

Let *Assumption SpecialDiff* hold. Then there exists a unique optimal strategy, i.e. there exists $\Theta^* \in \mathcal{A}_t(x)$ with

$$J(t, \delta, \Theta^*, \kappa) = \inf_{\Theta \in \mathcal{A}_t(x)} J(t, \delta, \Theta, \kappa).$$

Proof. Thanks to Lemma 2.4.2, we only need to prove existence. We start by showing that there exists a sequence of strategies $(\bar{\Theta}^n)$ that converges weakly to a strategy Θ^* and minimizes the costs J , i.e., $\lim_{n \rightarrow \infty} J(\bar{\Theta}^n) = \inf_{\Theta \in \mathcal{A}_t(x)} J(\Theta)$. We conclude by deducing that $\lim_{n \rightarrow \infty} J(\bar{\Theta}^n) = J(\Theta^*)$.

Let (Θ^j) be a minimizing sequence of J . Due to the Komlós Theorem in the form of Lemma 3.5 from Kabanov (1999), there exists a Cesaro convergent subsequence (Θ^{j_m}) . That is

$$\bar{\Theta}^n := \frac{1}{n} \sum_{m=1}^n \Theta^{j_m}$$

converges in $\mathcal{A}_t(x)$ to a strategy Θ^* in the following sense. For almost every ω , the measures $\bar{\Theta}^n(\omega)$ on $[t, T]$ converge weakly to the measure $\Theta^*(\omega)$. Equivalently, for $s \in [t, T]$ and almost every ω , we have $\lim_{n \rightarrow \infty} \bar{\Theta}_s^n = \Theta_s^*$ whenever $\Delta\Theta_s^* = 0$. Note that $\Theta_{T+}^* = x$ a.s. Moreover, $(\bar{\Theta}^n)$ is again a minimizing sequence, since J is convex.

It remains to show that Θ^* attains the infimum. Applying (2.33) yields

$$J(\bar{\Theta}^n) = \frac{1}{2} \mathbb{E} \left[\frac{(D_{T+}^n)^2}{K_T} - \frac{\delta^2}{\kappa} + \int_{[t, T]} \eta_s (D_s^n)^2 ds \right], \quad (2.34)$$

$$J(\Theta^*) = \frac{1}{2} \mathbb{E} \left[\frac{(D_{T+}^*)^2}{K_T} - \frac{\delta^2}{\kappa} + \int_{[t, T]} \eta_s (D_s^*)^2 ds \right]. \quad (2.35)$$

Based on Lemma 2.2.10, $\lim_{n \rightarrow \infty} D_s^n = D_s^*$ for almost every ω and every point $s \in [t, T+]$ of continuity of Θ^* . With Lebesgue's dominated convergence theorem and *Assumption SpecialDiff* ii), iii), it is obvious from (2.34) and (2.35) that $\lim_{n \rightarrow \infty} J(\bar{\Theta}^n) = J(\Theta^*)$. \square

2.4.2 Wait and buy region structure

Under *Assumption SpecialDiff*, we will now exploit the uniqueness of the optimal strategy to prove WR-BR structure. Proposition 2.4.5 treats the discrete time case, which is then transferred to continuous time in Proposition 2.4.9. The following lemma is essential.

Lemma 2.4.4. (*V* continuous in y).

Let *Assumption SpecialDiff* hold. Then $V(t, \cdot, \kappa)$ is continuous.

Proof. The continuity in $y = 0$ follows from the argument

$$U(t, 1, 0, \kappa) = 0 \leq U(t, 1, \epsilon, \kappa) \leq \left(1 + \frac{\kappa}{2}\epsilon\right) \epsilon \xrightarrow{\epsilon \rightarrow 0} 0.$$

For $y > 0$ and all sequences of positive numbers with $y^{1,j} \nearrow y$, $y^{2,j} \searrow y$, let us show

$$V_1 := \lim_{j \rightarrow \infty} V(t, y^{1,j}, \kappa) = \lim_{j \rightarrow \infty} V(t, y^{2,j}, \kappa) =: V_2.$$

It is clear that $V_1 \leq V_2$, since V is increasing in y . In the following, we prove $V_1 \geq V_2$.

Define the ratio $\psi_j := \frac{y^{2,j}}{y^{1,j}} \searrow 1$ and $\Theta^{i,j} := \Theta^*(t, 1, y^{i,j}, \kappa)$ for $i \in \{1, 2\}$, where Θ^* denotes the corresponding optimal strategy. Then,

$$V(t, y^{2,j}, \kappa) = U(t, 1, y^{2,j}, \kappa) = J(t, 1, \Theta^{2,j}, \kappa) \leq J(t, 1, \psi_j \Theta^{1,j}, \kappa) = V(t, y^{1,j}, \kappa) + \Delta J_j.$$

with

$$\Delta J_j := J(t, 1, \psi_j \Theta^{1,j}, \kappa) - J(t, 1, \Theta^{1,j}, \kappa).$$

The assertion follows since $\lim_{j \rightarrow \infty} \Delta J_j = 0$. This is due to (2.33),

$$D_s (\psi_j \Theta^{1,j}) - D_s (\Theta^{1,j}) \xrightarrow{j \rightarrow \infty} 0$$

a.s. for all $s \in [t, T+]$ and again Lebesgue's dominated convergence theorem together with *Assumption SpecialDiff* ii), iii). \square

Proposition 2.4.5. (Discrete time case: WR-BR structure).

Let *Assumption SpecialDiff* hold. Then the value function U^N has WR-BR structure.

Proof. According to Proposition 2.1.4 and 2.4.3, we only need to show that the optimal initial trade $\Delta \Theta_{t_n}^* (t_n, \delta, x, \kappa)$ is increasing in x . Thanks to the scaling in Lemma 1.3.1,

$$\Delta \Theta_{t_n}^* (t_n, \delta, x, \kappa) = \delta \Delta \Theta_{t_n}^* \left(t_n, 1, \frac{x}{\delta}, \kappa \right).$$

Due to the splitting argument from Lemma 2.1.5 and the uniqueness of the optimal strategy, $\Delta \Theta_{t_n}^* (t_n, 1, \cdot, \kappa)$ must be increasing and continuous apart from a possible discontinuity in the form of a jump back to zero. That is there might exist $y > 0$ with $\Delta \Theta_{t_n}^* (t_n, 1, y-, \kappa) > 0$ and $\Delta \Theta_{t_n}^* (t_n, 1, y+, \kappa) = 0$. In the following, we exclude such discontinuities using a Komlós argument as in the proof of Proposition 2.4.3, the continuity of V in y and the uniqueness.

Suppose for a contradiction that such a discontinuity exists in $y > 0$. Take the notation and monotonic sequences $(y^{i,j})$ from the proof of Lemma 2.4.4. Since V is continuous in y ,

$$J (t_n, 1, \Theta^{1,j}, \kappa) = V (t_n, y^{1,j}, \kappa) \xrightarrow{j \rightarrow \infty} V (t_n, y, \kappa).$$

Define $b_j := \frac{y}{y^{1,j}} \searrow 1$. As in the proof of Lemma 2.4.4, one shows that

$$J (t_n, 1, b_j \Theta^{1,j}, \kappa) - J (t_n, 1, \Theta^{1,j}, \kappa) \xrightarrow{j \rightarrow \infty} 0.$$

Therefore, $(b_j \Theta^{1,j})$ is a minimizing sequence, i.e., $b_j \Theta^{1,j} \in \mathcal{A}_{t_n}^N (y)$ and

$$\lim_{j \rightarrow \infty} J (t_n, 1, b_j \Theta^{1,j}, \kappa) = V (t_n, y, \kappa).$$

As in the proof of Proposition 2.4.3, we can take this minimizing sequence to define $\overline{\Theta}$ as the weak limit of the averaged sum over a subsequence of $(b_j \Theta^{1,j})$ such that $J(t_n, 1, \overline{\Theta}, \kappa) = V(t_n, y, \kappa)$. Due to the construction of $\overline{\Theta}$,

$$\Delta \overline{\Theta}_{t_n} (t_n, 1, y, \kappa) > 0.$$

Analogously, one constructs $\overline{\overline{\Theta}}$ from the sequence $\left(\frac{y}{y^{2,j}} \Theta^{2,j} \right)$ with initial trade

$$\Delta \overline{\overline{\Theta}}_{t_n} (t_n, 1, y, \kappa) = 0.$$

This is a contradiction to the uniqueness of the optimal strategy. \square

Remark 2.4.6. Notice that this argument does not hold for continuous trading time.

Having proved the WR-BR result for discrete time, let us now transfer it to continuous time in Proposition 2.4.9. To do so, we need Lemma 2.4.7 and Lemma 2.4.8.

Lemma 2.4.7. (Approximation via step functions).

Let Assumption SpecialDiff hold. For $\Theta \in \mathcal{A}_t(x)$, let $\Theta^N \in \mathcal{A}_t^{2^N}(x)$ be its approximation from below by an equidistant grid step function. More precisely, define $\mathcal{T}_t^0 := \{t, T\}$, $\mathcal{T}_t^{N+1} := \mathcal{T}_t^N \cup \{(s + \frac{T-t}{2^{N+1}}) \wedge T | s \in \mathcal{T}_t^N\}$ and

$$\Theta_s^N := \left\{ \begin{array}{ll} 0 & \text{if } s = t \\ \Theta_{u+} & \text{if } s \in (u, u + \frac{T-t}{2^N}] , u \in \mathcal{T}_t^N \\ x & \text{if } s = T+ \end{array} \right\}.$$

Then $J(t, 1, \Theta, \kappa) = \lim_{N \rightarrow \infty} J(t, 1, \Theta^N, \kappa)$.

Proof. Proceed as at the end of the proof of Proposition 2.4.3. That is we only need to show that Θ^N converges weakly to Θ . Due to $\mathcal{T}_t^N \subset \mathcal{T}_t^{N+1}$, Θ^N is increasing in N . For all $s \in [t, T+]$, the sequence $(\Theta_s^N)_{N \in \mathbb{N}}$ is bounded above by Θ_s . Hence, it is convergent. Due to the definition of Θ^N , we must even have $\lim_{N \rightarrow \infty} \Theta_s^N = \Theta_s$ for all $s \in [t, T]$ where $\Delta\Theta_s = 0$. \square

Lemma 2.4.8. (Cesaro weak convergence).

Fix $t \in [0, T]$, $\kappa \in (0, \infty)$ and for various $x \in [0, \infty)$ consider

$$(\Theta^N(t, 1, x, \kappa))_{N \in \mathbb{N}} \subset \mathcal{A}_t(x).$$

Then there exists a subsequence $N_j(t, \kappa)$, which does not depend on x , and a set of strategies $\tilde{\Theta}(t, 1, \cdot, \kappa)$ such that for all $x \in [0, \infty) \cap \mathbb{Q}$

$$\frac{1}{m} \sum_{j=1}^m \Theta^{N_j}(t, 1, x, \kappa) \xrightarrow{m \rightarrow \infty} \tilde{\Theta}(t, 1, x, \kappa). \quad (2.36)$$

Proof. Since \mathbb{Q} is countable, we can write $[0, \infty) \cap \mathbb{Q} = \{x_1, x_2, \dots\}$. For each $x \in [0, \infty)$, the Komlós Theorem guarantees the existence of a subsequence $N_j(t, x, \kappa)$ such that the desired weak convergence holds. That is we get $(N_j^{(1)})_{j \in \mathbb{N}} \subset \mathbb{N}$ for x_1 and extract the subsequence $N_j^{(2)}$ for x_2 from $N_j^{(1)}$ etc. The Cantor diagonal sequence $N_j := N_j^{(j)}$ then guarantees the Cesaro weak convergence of $\Theta^{N_j}(t, 1, x, \kappa)$ for all $x \in [0, \infty) \cap \mathbb{Q}$. \square

Proposition 2.4.9. (Continuous time: WR-BR structure).

Let Assumption SpecialDiff hold. Then the value function U has WR-BR structure.

Proof. As in the proof of Proposition 2.4.5, we only need to exclude the jump back to zero of $x \mapsto \Delta\Theta_t^*(t, 1, x, \kappa)$. Let $\Theta^N \in \mathcal{A}_t^{2^N}(x)$ be the approximation of $\Theta^* \in \mathcal{A}_t(x)$ by step functions from below as in Lemma 2.4.7. Then

$$J(t, 1, \Theta^*, \kappa) = \lim_{N \rightarrow \infty} J(t, 1, \Theta^N, \kappa).$$

Let Θ^{*N} be the unique optimal strategy within $\mathcal{A}_t^{2^N}(x)$, i.e.

$$J(t, 1, \Theta^N, \kappa) \geq J(t, 1, \Theta^{*N}, \kappa) \geq J(t, 1, \Theta^*, \kappa).$$

Hence,

$$J(t, 1, \Theta^*, \kappa) = \lim_{N \rightarrow \infty} J(t, 1, \Theta^{*N}, \kappa).$$

That is for each $x \in [0, \infty)$, $(\Theta^{*N}(t, 1, x, \kappa))_{N \in \mathbb{N}}$ is a minimizing sequence, and for each $N \in \mathbb{N}$, $x \mapsto \Delta \Theta_t^{*N}(t, 1, x, \kappa)$ is increasing thanks to Proposition 2.4.5.

Apply Lemma 2.4.8 to Θ^{*N} . Since the cost function is convex in the strategy and our optimal strategy is unique, the resulting $\tilde{\Theta}(t, 1, \cdot, \kappa)$ must coincide with $\Theta^*(t, 1, \cdot, \kappa)$ for all $x \in [0, \infty) \cap \mathbb{Q}$. Setting $\Theta_t^{*N_j}(t, 1, x, \kappa) := \Theta_{t+}^{*N_j}(t, 1, x, \kappa)$ and $\tilde{\Theta}_t(t, 1, x, \kappa) := \tilde{\Theta}_{t+}(t, 1, x, \kappa)$ does not disturb the corresponding weak convergence in (2.36). We then have a pointwise convergence at t such that the monotonicity of $x \mapsto \Theta_t^{*N_j}(t, 1, x, \kappa)$ transfers to $\tilde{\Theta}$ as desired. Since we only need to exclude the downward jump, it suffices to have this monotonicity on the rational numbers. \square

Example 2.4.10. Let K be deterministic such that $\eta_s \leq 0$ for all $s \in [t, T]$, e.g. $K_s = e^{-\alpha s}$ with $\alpha \geq 2\rho$. We can easily construct two different strategies $\Theta', \Theta'' \in \mathcal{A}_t(x)$ such that the corresponding deviation processes satisfy $D'_{T+} = D''_{T+}$. According to (2.33), J is not strictly convex for this choice of K and Proposition 2.4.9 does not apply. Nonetheless, we know from Subsection 2.2.3 that a unique optimal strategy exists and WR-BR structure holds in the deterministic case.

2.4.3 On the convexity assumptions

Under *Assumption SpecialDiff*, we proved existence and uniqueness of optimal strategies in Subsection 2.4.1 and WR-BR structure in Subsection 2.4.2. That is our results apply to positive diffusions K that satisfy the following three conditions.

Assumption. (Technical assumptions on K).

- i) $\eta_t := \frac{2\rho_t}{K_t} + \frac{\mu(t, K_t)}{K_t^2} - \frac{\sigma^2(t, K_t)}{K_t^3} > 0$ for all $t \in [0, T]$
- ii) $\mathbb{E} \left[\frac{\sup_{t \in [0, T]} K_t^2}{\inf_{t \in [0, T]} K_t} \right] < \infty$
- iii) $\mathbb{E} \left[\left(\int_0^T |\eta_t| dt \right) \left(\sup_{t \in [0, T]} K_t^2 \right) \right] < \infty$

We now aim to show that these are not merely abstract assumptions, but are indeed satisfied for some standard processes. We start with deterministic K and the time-homogeneous GBM, where we have already seen WR-BR results in Section 2.2 and 2.3. We could go ahead with any desired positive diffusion for K . As an example, we pick the Cox-Ingersoll-Ross (CIR) process. With its mean-reversion, it is particularly interesting from an economic point of view such that we also take the CIR process for our numerical illustrations in Section 3.3.

Proposition 2.4.11. (Deterministic case).

Take Assumption Smooth with $2\rho_t + \frac{K'_t}{K_t} > 0$ for all $t \in [0, T]$.
Then Assumption SpecialDiff holds.

Proof. Condition i) is equivalent to $2\rho_t + \frac{K'_t}{K_t} > 0$ and ii), iii) are clearly satisfied for deterministic, continuous K . \square

Proposition 2.4.12. (GBM case).

Take Assumption GBM with constant $\rho, \bar{\mu}, \bar{\sigma}$ and $2\rho + \bar{\mu} - \bar{\sigma}^2 > 0$.
Then Assumption SpecialDiff holds.

Proof. i) We have $\eta_t = \frac{1}{K_t} (2\rho + \bar{\mu} - \bar{\sigma}^2) > 0$. ii) Thanks to Hölder's inequality,

$$\mathbb{E} \left[\frac{(\sup_{t \in [0, T]} K_t)^2}{\inf_{t \in [0, T]} K_t} \right] \leq \mathbb{E} \left[\sup_{t \in [0, T]} K_t^4 \right]^{\frac{1}{2}} \mathbb{E} \left[\sup_{t \in [0, T]} q_t^2 \right]^{\frac{1}{2}}.$$

Using the explicit form of the GBM, $K_t = K_0 e^{\bar{\sigma} W_t^K + (\bar{\mu} - \frac{\bar{\sigma}^2}{2})t}$, yields

$$\mathbb{E} \left[\sup_{t \in [0, T]} K_t^4 \right] \leq K_0^4 \max \left\{ 1, e^{4(\bar{\mu} - \frac{\bar{\sigma}^2}{2})T} \right\} \mathbb{E} \left[\exp \left(4\bar{\sigma} \sup_{t \in [0, T]} W_t^K \right) \right].$$

This expression is finite due to the reflection principle. It says that $(\sup_{t \in [0, T]} W_t^K)$ has the same distribution as the absolute value of a Brownian motion at time T . The second expectation in the product above is finite, since $q_t = \frac{1}{K_t}$ is also a GBM with drift $(\bar{\sigma}^2 - \bar{\mu})$ and volatility $\bar{\sigma}$. iii) Due to the form of η_t , it is enough to consider

$$\mathbb{E} \left[\int_0^T \left(\sup_{t \in [0, T]} K_t \right)^2 \frac{1}{K_t} dt \right] \leq T \mathbb{E} \left[\frac{(\sup_{t \in [0, T]} K_t)^2}{\inf_{t \in [0, T]} K_t} \right].$$

This term is finite according to ii). \square

Proposition 2.4.13. (CIR case).

Take Assumption SpecialCIR. Then Assumption SpecialDiff holds.

Recall the Burkholder-Davis-Gundy Inequalities. See, e.g., Karatzas and Shreve (2000), page 166 for a proof.

The Burkholder-Davis-Gundy Inequalities.

For every $m > 0$, there exist universal positive constants k_m and K_m such that

$$k_m \mathbb{E} [\langle M \rangle_\tau^m] \leq \mathbb{E} \left[\left(\max_{t \leq \tau} |M_t| \right)^{2m} \right] \leq K_m \mathbb{E} [\langle M \rangle_\tau^m]$$

for every continuous local martingale M and every stopping time τ .

Proof of Proposition 2.4.13.

The CIR process stays a.s. strictly positive, as the Feller condition $2\bar{\mu}\bar{K} \geq \bar{\sigma}^2$ is met. Moreover, it turns out that $\eta_t = \frac{1}{K_t}(2\rho - \bar{\mu}) + \frac{1}{K_t^2}(\bar{\mu}\bar{K} - \bar{\sigma}^2) > 0$ under *Assumption SpecialCIR*. Condition ii) and iii) both hold by showing

$$\mathbb{E} \left[\frac{(\sup_{t \in [0, T]} K_t)^2}{(\inf_{t \in [0, T]} K_t)^2} \right] < \infty.$$

Thanks to Hölder's inequality,

$$\mathbb{E} \left[\frac{(\sup_{t \in [0, T]} K_t)^2}{(\inf_{t \in [0, T]} K_t)^2} \right] \leq \mathbb{E} \left[\sup_{t \in [0, T]} K_t^8 \right]^{\frac{1}{4}} \mathbb{E} \left[\sup_{t \in [0, T]} q_t^{\frac{8}{3}} \right]^{\frac{3}{4}}. \quad (2.37)$$

Since the drift of the CIR process is bounded above, we can isolate the local martingale part of K to use the Burkholder-Davis-Gundy Inequality. For positive constants \bar{c}_n ,

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} K_t^8 \right] &\leq \bar{c}_1 \left\{ (\bar{\mu}\bar{K}T)^8 + \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \bar{\sigma} \sqrt{K_s} dW_s^K \right|^8 \right] \right\} \\ &\leq \bar{c}_2 \left\{ (\bar{\mu}\bar{K}T)^8 + \mathbb{E} \left[\left(\int_0^T \bar{\sigma}^2 K_s ds \right)^4 \right] \right\}. \end{aligned} \quad (2.38)$$

This expression is finite, since all positive moments of the CIR process exist. See, e.g., Filipovic and Mayerhofer (2009).

It remains to show that the second term from the right hand side of (2.37) is finite. The order book height is an inverse CIR process and can be described by

$$dq_t = (\bar{\mu}q_t - (\bar{\mu}\bar{K} - \bar{\sigma}^2)q_t^2) dt - \bar{\sigma}q_t^{\frac{3}{2}} dW_t^K.$$

With these preparations, we can proceed analogously to (2.38)

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} q_t^{\frac{8}{3}} \right] &\leq \bar{c}_3 \left\{ \left(\frac{\bar{\mu}^2 T}{4(\bar{\mu}\bar{K} - \bar{\sigma}^2)} \right)^{\frac{8}{3}} + \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \bar{\sigma} q_s^{\frac{3}{2}} dW_s^K \right|^{\frac{8}{3}} \right] \right\} \\ &\leq \bar{c}_4 \left\{ \left(\frac{\bar{\mu}^2 T}{4(\bar{\mu}\bar{K} - \bar{\sigma}^2)} \right)^{\frac{8}{3}} + \mathbb{E} \left[\left(\int_0^T \bar{\sigma}^2 q_s^3 ds \right)^{\frac{4}{3}} \right] \right\}. \end{aligned} \quad (2.39)$$

We are done, since $\mathbb{E} \left[\left(\int_0^T q_s^3 ds \right)^{\frac{4}{3}} \right] \leq \bar{c}_5 \int_0^T \mathbb{E}[q_s^4] ds$ and the fourth moment of the inverse CIR process exists for $\bar{\mu}\bar{K} > 2\bar{\sigma}^2$. For an explicit calculation of the negative moments of the CIR process, see, e.g., Ahn and Gao (1999). \square

2.5 Counterintuitive trading regions

We have seen a variety of approaches how to prove WR-BR structure for different assumptions on K . Nevertheless, we did not succeed in showing this result in full generality. In this section, let us demonstrate the existence of situations where, when there is a large number of shares remaining to be purchased, we wait rather than buy, contrary to what might be expected. That is there are indeed examples with WR-BR-WR structure. First of all, let us consider three trading instances and successively present a binomial model, a CIR process and a time-inhomogeneous GBM that all lead to a WR-BR-WR example. Afterwards, a binomial model for K in continuous trading time is given. However, we could not work out a WR-BR-WR example for the CIR process or the time-inhomogeneous GBM in continuous time.

According to Proposition 2.2.4, WR-BR structure holds in full generality for the deterministic case. Therefore, K has to contain some kind of stochastics in order to produce a WR-BR-WR example. Moreover, the following proposition verifies that we need to have at least three trading instances. It shows that WR-BR structure always applies for two trading instances.

Proposition 2.5.1. (WR-BR structure for two trading instances).

Let $N = 1$, i.e. $0 = t_0 < t_1 = T$. Then the value function has WR-BR structure with

$$V^1(t_0, y, \kappa) = \frac{1}{2} \mathbb{E}_{t_0, \kappa}[K_T] y^2 + a_0 y - \left\{ \begin{array}{ll} \frac{[(\mathbb{E}_{t_0, \kappa}[K_T] - \kappa a_0) y - (1 - a_0)]^2}{2\kappa + 2\mathbb{E}_{t_0, \kappa}[K_T] - 4\kappa a_0} & \text{if } y > c(t_0, \kappa) \\ 0 & \text{otherwise} \end{array} \right\},$$

$$c(t_0, \kappa) = \left\{ \begin{array}{ll} \frac{1 - a_0}{\mathbb{E}_{t_0, \kappa}[K_T] - \kappa a_0} & \text{if } \mathbb{E}_{t_0, \kappa}[K_T] > \kappa a_0 \\ \infty & \text{otherwise} \end{array} \right\}.$$

Proof. We know that $U^1(t_1, \delta, x, \kappa) = (\delta + \frac{\kappa}{2}x)x$. The assertion follows from

$$U^1(t_0, \delta, x, \kappa) = \min_{\xi \in [0, x]} \left\{ \left(\delta + \frac{\kappa}{2}\xi \right) \xi + \mathbb{E}_{t_0, \kappa} [U^1(t_1, (\delta + \kappa\xi)a_0, x - \xi, K_T)] \right\}.$$

□

2.5.1 Binomial model in discrete time

In view of the discussions above, it is natural to construct a WR-BR-WR example for three trading instances $\{t_0, t_1, t_2\}$ and two states of the world $\Omega = \{\omega_A, \omega_B\}$ each being equally likely. To fully specify this *binomial model*, we can choose seven constants

$$a_0, a_1, \kappa_0, \kappa_1^A := K_{t_1}(\omega_A), \kappa_2^A := K_{t_2}(\omega_A), \kappa_1^B := K_{t_1}(\omega_B), \kappa_2^B := K_{t_2}(\omega_B).$$

Let us explain our parameter choice given in Figure 2.3. It guarantees $c(t_1, \kappa_1^A) = c(t_1, \kappa_1^B) = 1$, which simplifies our calculations. The main idea is to take model parameters such that the backward induction, that we did to prove WR-BR structure for the

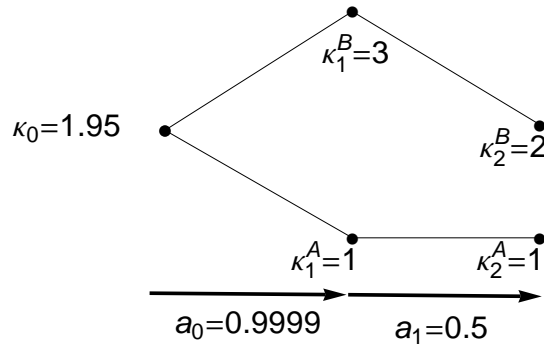


Figure 2.3: Seven constants that specify the binomial model with three trading instances.

deterministic case, does not work anymore. With dynamic programming (1.13),

$$U^2(t_0, \delta, x, \kappa_0) = \min_{\xi \in [0, x]} \tilde{U}^2(t_0, \delta, x, \kappa_0, \xi), \quad (2.40)$$

$$\tilde{U}^2(t_0, \delta, x, \kappa_0, \xi) := \left(\delta + \frac{\kappa_0}{2} \xi \right) \xi + \mathbb{E}_{t_0} [U^2(t_1, (\delta + \kappa_0 \xi) a_0, x - \xi, K_{t_1})],$$

where the exponents indicate that $N = 2$. We can compute this expectation using Proposition 2.5.1:

$$\begin{aligned} \mathbb{E}_{t_0} [V^2(t_1, y, K_{t_1})] &= \frac{1}{2} [V^2(t_1, y, \kappa_1^A) + V^2(t_1, y, \kappa_1^B)] \\ &= \frac{\kappa_2^A + \kappa_2^B}{4} y^2 + a_1 y - \frac{1}{2} \left\{ \begin{array}{ll} \sum_{C \in \{A, B\}} \frac{[(\kappa_2^C - \kappa_1^C a_1) y - (1 - a_1)]^2}{2\kappa_1^C + \kappa_2^C - 4\kappa_1^C a_1} & \text{if } y > 1 \\ 0 & \text{otherwise} \end{array} \right\} \end{aligned}$$

This term is piecewise quadratic in y with coefficients depending on $a_1, \kappa_1^A, \kappa_2^A, \kappa_1^B, \kappa_2^B$. For $y > 1$, our parameters ensure that inequalities (2.7) are violated. More precisely,

$$4\alpha\gamma + \beta - \beta^2 = -0.3125 < 0.$$

Scenario A is the liquid scenario and B corresponds to an illiquid market. We can hope to get a counterexample for this choice of parameters, but we still have to check this. To do so, we take different x and plot

$$\xi \mapsto \tilde{U}^2(t_0, 1, x, 1.95, \xi).$$

The result is given in Figure 2.4. When the total order is as small as $x = 0.9$, it is optimal not to do an initial trade. The transition from wait to buy region is approximately at $x = 0.95$. For $x = 1$, we are in the buy region and one optimally trades about two percent of the total order at time t_0 . But at $x = 5.75$, we switch from buy to wait region and stay in the wait region for all larger values of x . The graph for $x = 5.75$ nicely illustrates the *non-uniqueness* of the optimal strategy at the transition from buy to wait region.

To get the entire picture, one can now analyze the situation for different values of κ_0 . Figure 2.5 indicates for each point (κ_0, x) if it belongs to the buy or wait region. It is

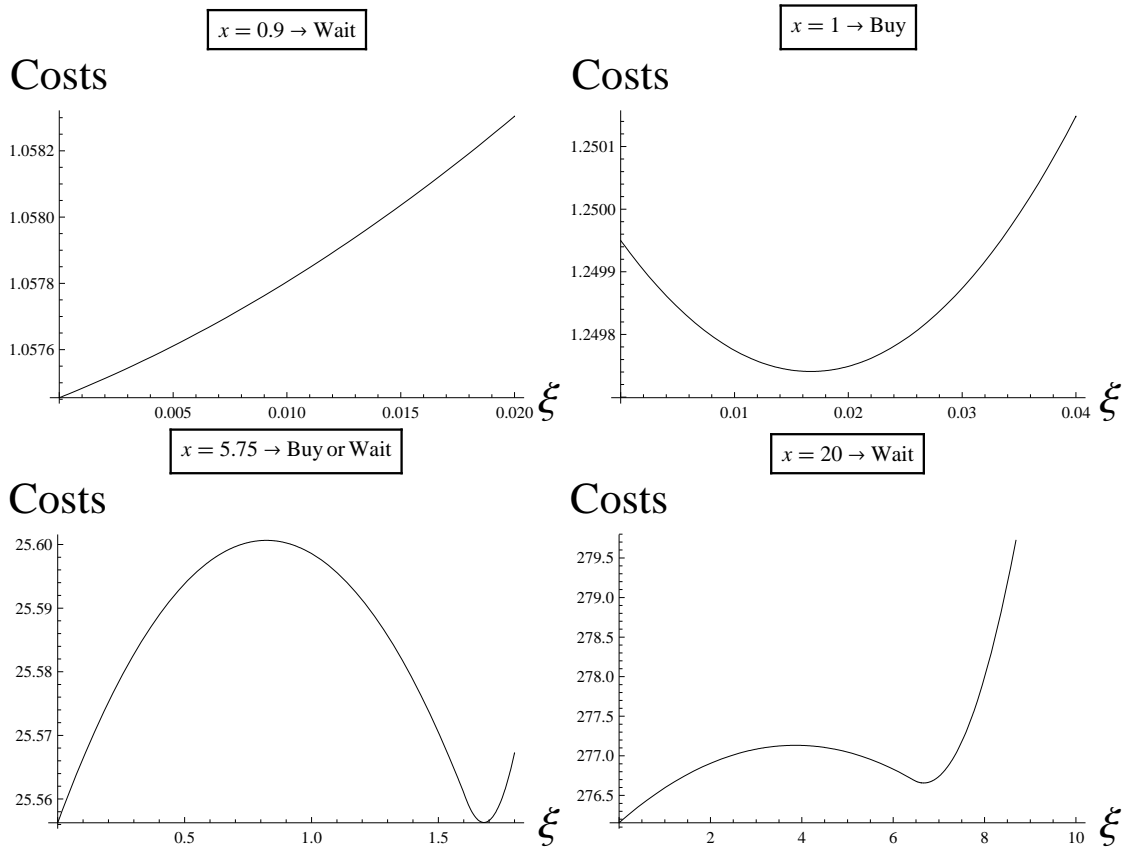


Figure 2.4: For the parameters from Figure 2.3 and total order size $x = 0.9, 1, 5.75, 20$, the graphs plot the dependence of the costs $\tilde{U}^2(t_0, 1, x, 1.95, \xi)$ on the initial trade ξ .

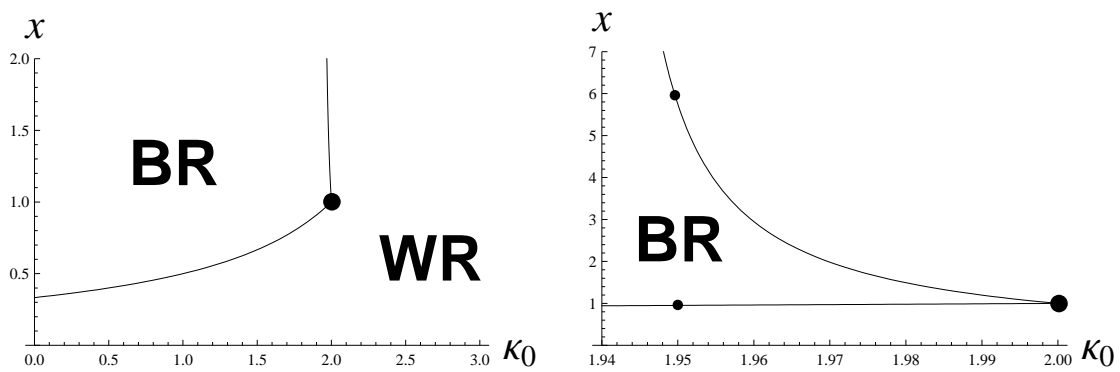


Figure 2.5: For the parameters from Figure 2.3, but different values of κ_0 , we illustrate the wait and buy region. Looking more closely at the large dot $(\kappa_0, x) = (2, 1)$ yields the picture on the right-hand side. The buy region has the shape of a wedge.

created by computing the optimal initial trade $\xi(\kappa_0, x)$ of $\tilde{U}^2(t_0, 1, x, \kappa_0, \xi)$ analytically. WR-BR-WR structure occurs for $\kappa_0 \in (1.94, 2)$. The upper barrier from buy to wait region has an asymptote at $\kappa_0 = 1.94$. For the case $\kappa_0 = 1.95$ that we discussed in Figure 2.4, the small dots on the right-hand side of Figure 2.5 point out the transitions from wait to buy region and buy to wait region respectively. For expensive $\kappa_0 \geq 2$, we are not trading irrespectively of the size of the total order. For inexpensive $\kappa_0 \leq 1.94$, we have the usual WR-BR situation. On the interval in between, the large investor has an incentive not to trade for large positions x . The resilience between t_0 and t_1 is extremely low and waiting until t_1 has the advantage of gaining information whether scenario A or B has occurred. That is there is a tradeoff between gaining information by waiting until the next time instance and attracting resilience by trading right now.

2.5.2 Cox-Ingersoll-Ross process, discrete time

The question arises if the toy example of the last section is totally artificial. Does it still work for more realistic models like the CIR process or the time-inhomogeneous GBM? Does it still work when trading is allowed in continuous time? The answer to both questions is *yes*. We tackle the first one in this subsection by taking CIR parameters which are inspired by the binomial model above. We can then confirm WR-BR-WR structure for three trading instances. In the next subsection, the same thing is done for the GBM. The second question is addressed in the last subsection by analyzing a binomial model in continuous trading time.

Let us consider a CIR process $dK_s = \bar{\mu}(\bar{K} - K_s) ds + \bar{\sigma}\sqrt{K_s}dW_s^K$, constant resilience and trading times $\{t_0, t_1, t_2\}$ with $t_0 = 0$. That is we need to specify six constants

$$\begin{aligned} t_1 &= 0.0072, \quad t_2 = 1.0072, \quad \rho = 1.3863, \\ \bar{\mu} &= 0.6931, \quad \bar{K} = 1, \quad \bar{\sigma} = 5.2523. \end{aligned} \tag{2.41}$$

For this choice, *Assumption SpecialCIR* is not satisfied. That is we were not able to prove WR-BR structure. There are some similarities to the binomial model from above. E.g. the high volatility makes illiquid scenarios with $K_t \gg \bar{K}$ likely to occur. Also the other parameters are chosen to tendentially reproduce the toy model with

$$e^{-\rho t_1} \approx 0.99, \quad e^{-\rho(t_2 - t_1)} \approx 0.25, \quad \mathbb{E}_{t_0, \kappa_0=2.005}[K_{t_1}] = \mathbb{E}_{t_1, K_{t_1}=3}[K_{t_2}] = 2.$$

It turns out that this CIR model leads to a WR-BR-WR example. With Proposition 2.5.1 and taking the density function of the CIR process together with a numerical integration, we can compute $\tilde{U}^2(t_0, 1, x, \kappa_0, \xi)$ from (2.40). For each point (κ_0, x) , we calculate the costs for different trades ξ from an equidistant grid $\{0, d\xi, \dots, x\}$. The point (κ_0, x) belongs to the wait region if and only if the costs for $\xi = 0$ are smaller than the costs on the remaining grid. Doing this procedure for several points (κ_0, x) yields Figure 2.6.

As for the binomial model, there exist choices of κ_0 that lead to WR-BR-WR structure. But instead of a wedge-shaped buy region, we get a tongue-shaped upper wait region,

which is located around the mean-reversion level $\bar{K} = 1$. This is a qualitative difference. That is for large values of κ_0 , we do not have a pure wait region anymore.

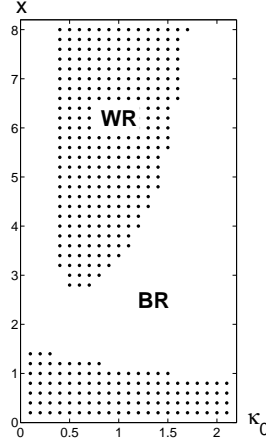


Figure 2.6: This figure shows a WR-BR-WR example for the CIR process with parameters (2.41) and three trading instances. Points $(\kappa_0, x) \in \{0.1, 0.2, \dots, 2.1\} \times \{0.2, 0.4, \dots, 8\}$ are considered. The wait region is shaded black.

2.5.3 Geometric Brownian motion, discrete time

Let us consider a GBM $dK_s = K_s(\bar{\mu}_s ds + \bar{\sigma} dW_s^K)$ with time-inhomogeneity in the drift, constant resilience and $\{t_0, t_1, t_2\}$ with $t_0 = 0$. That is we need to specify six constants

$$\begin{aligned} t_1 &= 0.0072, \quad t_2 = 1.0072, \quad \rho = 1.3863, \\ \tilde{\mu}_1 &= -1.5, \quad \tilde{\mu}_2 = -1.35, \quad \bar{\sigma} = 5.2523 \end{aligned} \quad (2.42)$$

with $\bar{\mu}_s \equiv \tilde{\mu}_1$ on $[t_0, t_1]$ and $\bar{\mu}_s \equiv \tilde{\mu}_2$ on $(t_1, t_2]$. Let us motivate this choice.

According to Proposition 2.5.1, $\tilde{\mu}_2 \leq -\rho$ yields $\tilde{c}(t_1) = \infty$. That is we would only consider two trading instances $\{t_0, t_2\}$. But with two trading instances, we cannot construct a WR-BR-WR example. Therefore we have to choose $\tilde{\mu}_2 > -\rho$. We also need to have $\tilde{\mu}_1 < -\rho$. Otherwise, Proposition 2.3.1 would apply. So we can fix

$$\tilde{\mu}_1 < -\rho \quad \text{and} \quad \tilde{\mu}_2 > -\rho. \quad (2.43)$$

Then we see from the proof of Proposition 2.3.1 b) that

$$\lim_{z \rightarrow \infty} \frac{\partial}{\partial z} \mathcal{B}^W(W^{WR})(t_0, z) > 0.$$

Hence, large values of $z = \frac{\kappa_0 x}{\delta}$ must be in the WR. That is we either have

- i) WR-BR-WR structure or

ii) we optimally never trade at t_0 irrespective of z .

Let us point out how i) is possible, although $\tilde{\mu}_1 < -\rho$ strongly suggests ii). Compare two strategies $\Theta, \Theta' \in \mathcal{A}_0^2(x)$ described by the trades $(\xi_0, \xi_1, \xi_2), (0, \xi_0 + \xi_1, \xi_2)$. Heuristically, if only case ii) would occur, this would suggest $J(\Theta) - J(\Theta') \geq 0$. So let us check if this difference can become negative. Define

$$\begin{aligned} I_0 &:= \delta + \frac{\kappa_0}{2}\xi_0 - \delta e^{-\rho t_1} - \frac{\mathbb{E}_{t_0, \kappa_0}[K_{t_1}]}{2}\xi_0 \geq 0, \\ I_1 &:= \mathbb{E}_{t_0, \kappa_0} [(\kappa_0 e^{-\rho t_1} - K_{t_1}) \xi_1], \\ I_2 &:= \mathbb{E}_{t_0, \kappa_0} [(\kappa_0 e^{-\rho t_2} - K_{t_1} e^{-\rho(t_2-t_1)}) \xi_2]. \end{aligned}$$

Then

$$J(0, \delta, \Theta, \kappa_0) - J(0, \delta, \Theta', \kappa_0) = (I_0 + I_1 + I_2) \xi_0.$$

For ξ_1, ξ_2 deterministic or independent of K_{t_1} , we would directly get $I_1, I_2 \geq 0$ and therefore $J(\Theta) - J(\Theta') \geq 0$. But we know from above that the optimal trade

$$\xi_1^* = \max\{0, \tilde{\xi}_1^*\} \quad \text{with} \quad \tilde{\xi}_1^* = \frac{x - \xi_0 - \frac{\tilde{c}(t_1)}{K_{t_1}}(\delta + \kappa_0 \xi_0) e^{-\rho t_1}}{1 + \tilde{c}(t_1)}$$

is passive in the liquidity. That is the scenarios ω with high $K_{t_1}(\omega)$ get a high weighting $\xi_1^*(\omega)$. Heuristically, small values of t_1 and large $(t_2 - t_1)$ make I_0 and I_2 negligible such that we can concentrate on I_1 . Plugging $\xi_1 = \tilde{\xi}_1^*$ into I_1 yields

$$I_1 = \frac{(x - \xi_0)\kappa_0 (e^{-\rho t_1} - e^{\tilde{\mu}_1 t_1})}{1 + \tilde{c}(t_1)} + \frac{\tilde{c}(t_1)}{1 + \tilde{c}(t_1)} (\delta + \kappa_0 \xi_0) e^{-\rho t_1} \left(1 - \kappa_0 e^{-\rho t_1} \mathbb{E}_{t_0, \kappa_0} \left[\frac{1}{K_{t_1}} \right] \right).$$

Since $\frac{1}{K_t}$ for $t \in [t_0, t_1]$ is a GBM with drift $(\bar{\sigma}^2 - \tilde{\mu}_1)$, we get

$$\mathbb{E}_{t_0, \kappa_0} [K_{t_1}^{-1}] = \kappa_0^{-1} \exp(\bar{\sigma}^2 - \tilde{\mu}_1).$$

Therefore, I_1 and in turn $J(\Theta) - J(\Theta')$ gets negative, as soon as we choose $\bar{\sigma}$ large enough. According to Proposition 2.5.1, we have $\tilde{c}(t_1) \rightarrow \infty$ as $\tilde{\mu}_2 \searrow -\rho$. That is the first summand in I_1 is positive, but gets small for $\tilde{\mu}_1 \approx \tilde{\mu}_2 \approx -\rho$. In summary, parameter choices with small t_1 , large $t_2, \bar{\sigma}$ and $\tilde{\mu}_1 \approx \tilde{\mu}_2 \approx -\rho$ satisfying condition (2.43) are likely to produce a WR-BR-WR example.

We check this for our parameter choice (2.42). For various $x \in [0, \infty)$, consider how the costs (2.40) depend on the initial trade

$$\xi \mapsto \tilde{U}^2(t_0, 1, x, 1, \xi).$$

Thereby, K_{t_1} is log-normally distributed and using Proposition 2.5.1, the expectation within \tilde{U}^2 can be computed by a numerical integration. It turns out that the optimal

initial trade is strictly positive for roughly $x \in (108, 778)$ which is the buy region. Smaller and larger values of x lie in the wait region.

We finally succeed in constructing a WR-BR-WR example in discrete trading time for the time-inhomogeneous GBM. Remember that we can do both dimension reductions in case of the GBM. That is the presented example leads to WR-BR-WR structure for each initial value κ_0 of K and the buy region is tubular in κ_0 .

2.5.4 Binomial model in continuous time

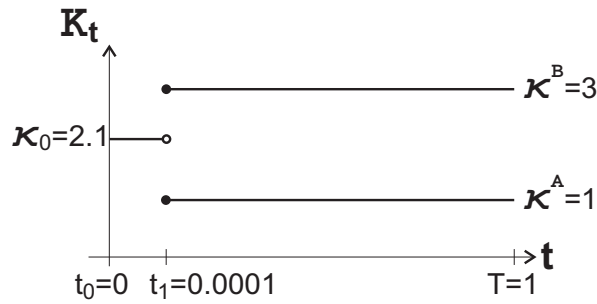


Figure 2.7: Binomial model in continuous time with two equally likely scenarios. Set $\rho \equiv 2$.

The construction of a WR-BR-WR example is still possible when trading is allowed in continuous time instead of being restricted to three trading instances. In order to show this analytically, we take again a simple binomial model with $\Omega = \{\omega_A, \omega_B\}$ and $\mathbb{P}[\{\omega_A\}] = \frac{1}{2}$. The process K stays constant on $[t_0, t_1)$ and on $[t_1, T]$ as illustrated in Figure 2.7. This situation might correspond to an announcement at t_1 , which can make the liquidity jump up or down. As in Proposition 2.2.12, one can show the existence of an optimal strategy, since Ω is countable.

We show in Lemma 2.5.2 that for $D_0 = 0$, it is optimal *not* to trade on $[0, t_1)$. That is WR-BR structure could only hold if $WR_t(\kappa_0) = [0, \infty)$ on $[0, t_1)$. But it turns out in Lemma 2.5.3 that there exists $t \in [0, t_1)$ such that $2 \in Br_t(\kappa_0)$. Hence, we cannot have WR-BR structure. For the proofs of these lemmata, it is not clear how to explicitly compute the value function at t_0 . Therefore, we exploit the constant K case from Proposition 2.2.3 to fully specify the value function at t_1 . This way, it is possible to find helpful lower and upper bounds of the value function at t_0 .

Lemma 2.5.2. *For the model stated in Figure 2.7 and all $x \in [0, \infty)$,*

$$U(t_0, 0, x, \kappa_0) = \mathbb{E}_{t_0} [U(t_1, 0, x, K_{t_1})].$$

Proof. The optimal strategy must consist of purchasing $\tilde{x} \in [0, x]$ shares on $[0, t_1)$ and the remaining order on $[t_1, T]$. When we look at each time interval separately and imagine that the trading strategy on $[0, t_1)$ would contain a discrete trade at t_1 , it

must be optimal to apply the constant K optimal strategy and value function \bar{U} as given in Proposition 2.2.3 on each interval. Then $D_{t_1} \geq \kappa_0 \tilde{x} e^{-\rho t_1}$ and accordingly

$$\begin{aligned} U(t_0, 0, x, \kappa_0) &\geq \min_{\tilde{x} \in [0, x]} \bar{U}(T - t_1, 0, \tilde{x}, \kappa_0) + \mathbb{E}_{t_0} [U(t_1, \kappa_0 \tilde{x} e^{-\rho t_1}, x - \tilde{x}, K_{t_1})] \\ &= \min_{\tilde{x} \in [0, x]} \frac{\kappa_0 \tilde{x}^2}{\rho t_1 + 2} + \frac{1}{2} [\bar{U}(t_1, \kappa_0 \tilde{x} e^{-\rho t_1}, x - \tilde{x}, \kappa^A) + \bar{U}(t_1, \kappa_0 \tilde{x} e^{-\rho t_1}, x - \tilde{x}, \kappa^B)]. \end{aligned}$$

For various $x \in [0, \infty)$, we can plot this term as a function of \tilde{x} . It turns out that it is minimal for $\tilde{x} = 0$. For $x = 1$, this is exemplarily shown on the left in Figure 2.8. \square

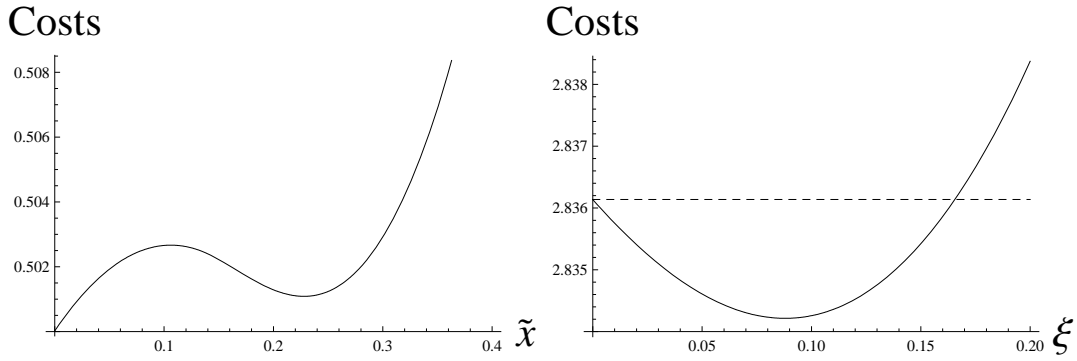


Figure 2.8: Lemma 2.5.2: Cost dependence on \tilde{x} for the binomial model and $x = 1$. Lemma 2.5.3: Cost dependence on ξ for the binomial model and $\delta = 1, x = 2$. The dashed line displays the expected optimal costs when there is no trading on $[0, t_1)$.

Lemma 2.5.3. *For the model stated in Figure 2.7, there exists $t \in [0, t_1)$ such that*

$$2 \in Br_t(\kappa_0).$$

Proof. We need to show that the value function is strictly smaller than the expected optimal costs when there is no trading on $[0, t_1)$, i.e.,

$$U(t_0, 1, 2, \kappa_0) < \mathbb{E}_{t_0} [U(t_1, e^{-\rho t_1}, 2, K_{t_1})]. \quad (2.44)$$

Define $\Theta(\xi)$ via $\Delta\Theta_0(\xi) = \xi$, $d\Theta_s(\xi) = \frac{\rho}{\kappa_0}(1 + \kappa_0\xi)ds$ with $\xi \leq \frac{2\kappa_0 - \rho}{\kappa_0(1+\rho)} \approx 0.35$. Then $\Theta(\xi) \in \mathcal{A}_0(2)$ and the corresponding D stays constant on $(0, t_1)$. Consequently,

$$\begin{aligned} U(t_0, 1, 2, \kappa_0) &\leq \min_{\xi} \left(1 + \frac{\kappa_0}{2}\xi\right) \xi + \int_0^{t_1} (1 + \kappa_0\xi) d\Theta_s(\xi) \\ &\quad + \mathbb{E}_{t_0} \left[U \left(t_1, 1 + \kappa_0\xi, 2 - \xi - \int_0^{t_1} d\Theta_s(\xi), K_{t_1} \right) \right]. \end{aligned}$$

As in the proof of Lemma 2.5.2, we can state the expectation in terms of \bar{U} from Proposition 2.2.3 and plot the entire term as a function of ξ . As desired, it turns out that there exists $\xi^* \in [0, 0.35]$ such that the corresponding costs are strictly lower than the expected optimal costs when there is no trading on $[0, t_1)$. See the right-hand side of Figure 2.8. \square

Remark 2.5.4. As mentioned in Subsection 2.2.1, the optimal deviation process for constant K is constant in time. This leads to the choice of the strategy $\Theta(\xi)$ in Lemma 2.5.3 and the fact that the corresponding costs are close enough to the value function. Also notice that the two lines on the right-hand side of Figure 2.8 are not equal for $\xi = 0$, since $d\Theta_s(0) = \frac{\rho}{\kappa_0} ds$ is strictly positive. The costs for $\xi = 0$ are slightly smaller than not trading on $[0, t_1)$.

We have now shown the existence of a WR-BR-WR example in continuous time for a specific choice of κ_0 in the binomial model. There exists $t \in [0, t_1)$ such that the barrier from wait to buy region lies in $(0, 2)$ and the barrier from buy to wait region in $(2, \infty)$. But we neither know t , nor the exact values of the barriers. Taking $t_1 \searrow t_0$, we get a model where trading is allowed for the impact coefficient κ_0 at t_0 and K immediately jumps to either κ^A or κ^B afterwards. It then becomes quite easy to numerically calculate the two barriers for different values of κ_0 . As in Section 2.5.1, this leads to a wedge-shaped buy region at t_0 . In particular, it is optimal not to trade for large values of κ_0 irrespective of the choice of x .

Ideally, one would like to know if a WR-BR-WR example in continuous time can also exist for more realistic K like the CIR process. Although we are not able to show WR-BR structure in full generality for the CIR process, we have not found such an example using our numerical scheme.

Chapter 3

Numerical scheme

The previous chapter analytically investigated the structure of solutions to our optimal execution problem. Let us now numerically compute the shape of the buy and wait region, the value function and corresponding optimal strategies. To do so, we follow the well-established Markov chain approximation method introduced by Kushner and his co-authors. An introduction to the method is given in Kushner and Dupuis (2001). For our problem, this yields a numerical scheme which is investigated in this chapter. The convergence proof for our scheme is closely related to Kushner and Martins (1991) and Budhiraja and Ross (2007). Similar to Davis and Norman (1990), but with general continuous utility functions, the paper of Budhiraja and Ross (2007) deals with the problem of optimal consumption and portfolio selection with proportional transaction costs, where an investor maximizes his expected discounted utility of consumption.

The HJB equation for our singular control problem is not elliptic. Moreover, as a variational inequality it is highly non-linear. To the best of our knowledge, there is no systematic approach that treats numerical schemes for these variational inequalities in several dimensions directly. Therefore, we use Kushner's method as an alternative to a verification argument. Instead of the HJB equation, the control problem itself is approximated. In other words, the value function is approximated on a grid and the state space dynamics are replaced by transition probabilities between the grid points. Thereby a finite difference scheme of the HJB equation can be used as guidance how to choose these transition probabilities consistently with the original state dynamics. In the following, we are going to show that this approximated value function converges to the original one as the grid size decreases to zero. The proof is by probabilistic methods only, and it turns out that the positivity of the transition probabilities, see (3.8), is the only assumption for the convergence result. The essential steps in the proof are to truncate the state space and to use tightness results. Due to our control problem being singular, it is necessary to do a time rescaling. Although the HJB equation is not used in the proof at all, we see in Section 3.2.2 that the numerical scheme resulting from the Markov chain method is equivalent to the implementation of the HJB equation by a finite difference scheme.

There are some peculiarities that make our problem more complicated than in Kushner and Martins (1991) and Budhiraja and Ross (2007). Therefore, it is a priori not clear if the Markov chain numerical scheme also converges in our case. In particular, we cannot just replace parts of the proof from the two mentioned papers. Instead, we have to set up the entire convergence proof with suitable adaptations for our problem.

Specifically, both papers analyze an infinite horizon singular control problem with two-dimensional state space and the control effects the state space dynamics always in the direction of two fixed vectors. By contrast, our problem has a finite time horizon, a three-dimensional state space, and most notably the control direction depends on K , i.e., it is state space dependent. This is for example relevant when we introduce the notion of simple strategies in Definition 3.1.18. Moreover, our cost structure is qualitatively different, since the costs contain an integral with respect to the control with the integrand being state space and control dependent. This, e.g., complicates the state space truncation, since the mentioned integrand is unbounded. At some places, we thus decide not to follow the methods presented in Kushner and Martins (1991) and Budhiraja and Ross (2007), but utilize the specific features of our optimal execution problem. The proof of Lemma 3.1.15 is an example for this.

3.1 Markov chain method

We start by giving an equivalent, but slightly reformulated version of our singular control problem. This is necessary in order to make the representation of our problem more similar to the one given in Budhiraja and Ross (2007). As before, Θ denotes the control, but instead of $\mathcal{A}_t(x)$ we consider the control set

$$\tilde{\mathcal{A}}_t := \{\Theta \in \mathcal{A}_t \mid \Theta_T = \Theta_{T+}\}$$

and incorporate the final jump trade in the cost functional (see below). Let *Assumption HomogDiff* hold. We typically think of a GBM or CIR process, which both satisfy this assumption. We exclude time-dependent drift and volatility here in order to get a time-homogeneous Markov chain approximation. This makes the convergence proof notationally simpler, although it should still work for time-inhomogeneous diffusions. For $\Theta \in \tilde{\mathcal{A}}_t$, let us recapitulate the state space dynamics

$$\begin{aligned} dD_s &= -\rho_s D_s ds + K_s d\Theta_s, \\ dX_s &= -d\Theta_s, \\ dK_s &= \mu(K_s) ds + \sigma(K_s) dW_s^K. \end{aligned}$$

Due to *Assumption HomogDiff*, K is non-negative. Set $Z_s := (D_s, X_s, K_s) \in [0, \infty) \times (-\infty, \infty) \times [0, \infty)$ for $s \in [t, T]$ and $Z_t := z := (\delta, x, \kappa) \in [0, \infty)^3$ as initial condition. (This three-dimensional z should not be mixed up with $z = \frac{\kappa x}{\delta}$ as introduced in Table 1.1.) Define the stopping time

$$\tau_t := \inf \{s \geq t \mid X_s \leq 0\} \wedge T.$$

It indicates the time when all shares are bought. Introduce the costs

$$J(t, z, \Theta) := \mathbb{E}_{t,z} \left[\int_{[t, \tau_t)} \alpha(Z_s, \Delta\Theta_s) d\Theta_s + g(Z_{\tau_t}) \right]$$

$$\alpha(z, \vartheta) := \delta + \frac{\kappa}{2} \vartheta, \quad g(z) := \left(\delta + \frac{\kappa}{2} x \right) x$$

Slightly abusing notation, our value function can be expressed as

$$U(t, z) = \inf_{\Theta \in \mathcal{A}_t(x)} J(t, \delta, \Theta, \kappa) = \inf_{\Theta \in \tilde{\mathcal{A}}_t} J(t, z, \Theta).$$

That is compared to the original formulation, we replace the constraint $\Theta_{T^+} = x$ by the stopping time τ_t and the additional cost term g . Notice that the two corresponding functions J are notationally only differentiated by the number of input parameters.

3.1.1 State space truncation

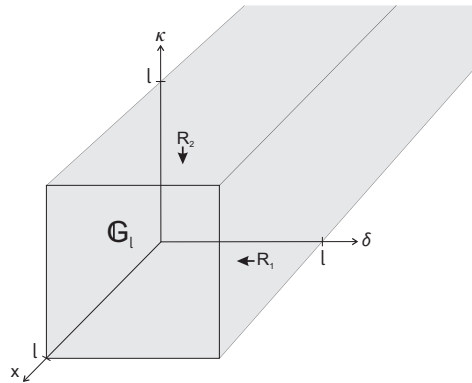


Figure 3.1: Illustration of the state space truncation using reflection.

In order to solve the control problem numerically, we need to restrain the state space process Z to some bounded set \mathbb{G} . One can either work with absorbing or reflecting boundaries of \mathbb{G} . We choose the second alternative and introduce a box \mathbb{G} which has some reflecting boundaries. Proposition 3.1.1 then proves that the truncated value function converges to the original one when enlarging the box. This result holds irrespective of the numerical scheme that we choose to approximate the value function.

For $l \in \mathbb{R}_{>0}$, define the box

$$\mathbb{G}_l := [0, l] \times (-\infty, l] \times [0, l].$$

The aim is to find a minimal reflection process such that Z stays in this box as illustrated in Figure 3.1. Since X is decreasing for $\Theta \in \tilde{\mathcal{A}}_t$ and the processes D, K stay

positive, we only need to bound D, K from above. Thus, for $s \in [t, T]$ we are looking for modified state dynamics

$$\begin{aligned} Z_l(s) &= (D_l(s), X_l(s), K_l(s)) \in \mathbb{G}_l \\ dD_l(s) &= -\rho_s D_l(s) ds + K_l(s) d\Theta_s - dR_1(s), \\ dX_l(s) &= -d\Theta_s, \\ dK_l(s) &= \mu(K_l(s)) ds + \sigma(K_l(s)) dW_s^K - dR_2(s) \end{aligned} \quad (3.1)$$

with $Z_l(t) = z = (\delta, x, \kappa) \in [0, l]^3$. More precisely, the reflection proces $R = (R_1, 0, R_2)$ should be componentwise nondecreasing, adapted and càglàd such that

$$\int_{[t, \infty)} \mathbb{I}_{\{D_l(s+) < l\}} dR_1(s) = \int_{[t, \infty)} \mathbb{I}_{\{K_l(s+) < l\}} dR_2(s) = 0.$$

According to Dupuis and Ishii (1991), this *Skorokhod problem* with normal reflection has a unique solution and the associated *Skorokhod map* is Lipschitz continuous with respect to the supremum norm. Therefore, the *state space truncated control problem* is well defined:

$$\begin{aligned} U_l(t, z) &:= \inf_{\Theta \in \tilde{\mathcal{A}}_t} J_l(t, z, \Theta), \\ J_l(t, z, \Theta) &:= \mathbb{E}_{t, z} \left[\int_{[t, \tau_t^l)} \alpha(Z_l(s), \Delta\Theta_s) d\Theta_s + g(Z_l(\tau_t^l)) \right], \\ \tau_t^l &:= \inf \{s \geq t \mid X_l(s) \leq 0\} \wedge T. \end{aligned} \quad (3.2)$$

The only difference between the stopping times τ_t and τ_t^l is that $X_l(t)$ needs to be smaller than l . Let us now show that U_l converges to U .

Proposition 3.1.1. (Convergence of state space truncation).

The function U_l converges locally uniformly to U on $[0, T] \times [0, \infty)^3$ as $l \rightarrow \infty$.

Proof. Let $\mathbb{G} \subset [0, \infty)^3$ be compact. The aim is to prove that for each $\tilde{\delta} > 0$, there exists $l_0 > 0$ such that for all $l > l_0$ and $(t, z) \in [0, T] \times \mathbb{G}$

$$|U_l(t, z) - U(t, z)| \leq \tilde{\delta}.$$

We start by showing that for all $l > l_0(\tilde{\delta}) > \hat{l}_0 := \max\{\delta \vee x \vee \kappa \mid (\delta, x, \kappa) \in \mathbb{G}\}$,

$$U_l(t, z) \leq U(t, z) + \tilde{\delta}. \quad (3.3)$$

Here, we still have to choose $l_0(\tilde{\delta})$ independent of (t, z) . Changing the roles of U_l and U in the following argument, we analogously get

$$U(t, z) \leq U_l(t, z) + \tilde{\delta}. \quad (3.4)$$

The assertion then follows from (3.3) and (3.4).

Let us prove (3.3). For $\epsilon > 0$, there exists an ϵ -optimal strategy $\Theta_{t,z}^\epsilon \in \tilde{\mathcal{A}}_t$ such that

$$J(t, z, \Theta^\epsilon) \leq U(t, z) + \epsilon. \quad (3.5)$$

For this strategy, define the events

$$A_{l,l_0}^{t,\epsilon} := \left\{ \omega \in \Omega \left| \sup_{s \in [t, \tau_t^l]} [D_l^\epsilon(s) + K_l(s)] \geq l_0 \right. \right\}.$$

For notational convenience, the dependence on z of this set and the involved process is not made precise, since this is not crucial in the sequel. Consider $A_{l,l_0}^{t,\epsilon}$ and its complement separately to get the estimate

$$\begin{aligned} U_l(t, z) \leq J_l(t, z, \Theta^\epsilon) &= \mathbb{E}_{t,z} \left[\mathbb{I}_{(A_{l,l_0}^{t,\epsilon})^c} \left\{ \int_{[t, \tau_t^l]} \alpha(Z_l^\epsilon(s), \Delta\Theta_s^\epsilon) d\Theta_s^\epsilon + g(Z_l^\epsilon(\tau_t^l)) \right\} \right] \\ &+ \mathbb{E}_{t,z} \left[\mathbb{I}_{A_{l,l_0}^{t,\epsilon}} \left\{ \int_{[t, \tau_t^l]} \alpha(Z_l^\epsilon(s), \Delta\Theta_s^\epsilon) d\Theta_s^\epsilon + g(Z_l^\epsilon(\tau_t^l)) \right\} \right] \\ &\leq J(t, z, \Theta^\epsilon) + \tilde{\delta}. \end{aligned} \quad (3.6)$$

Indeed, the first expectation in (3.6) is dominated by $J(t, z, \Theta^\epsilon)$, since Z_l^ϵ and Z^ϵ coincide on $(A_{l,l_0}^{t,\epsilon})^c$. It only remains to show that the second expectation in (3.6) is dominated by $\tilde{\delta}$. From (3.5) and (3.6), (3.3) follows by choosing ϵ small enough.

Define

$$\hat{K} := \sup_{s \in [0, T]} K(s).$$

We can always trade everything at once and so the costs are pathwise bounded by

$$\left[(\hat{l}_0 + \hat{K}\hat{l}_0) + \frac{\hat{K}}{2}\hat{l}_0 \right] \hat{l}_0 = \hat{l}_0^2 \left(1 + \frac{3}{2}\hat{K} \right).$$

Hence,

$$\begin{aligned} &\mathbb{E}_{t,z} \left[\mathbb{I}_{A_{l,l_0}^{t,\epsilon}} \left\{ \int_{[t, \tau_t^l]} \alpha(Z_l^\epsilon(s), \Delta\Theta_s^\epsilon) d\Theta_s^\epsilon + g(Z_l^\epsilon(\tau_t^l)) \right\} \right] \\ &\leq \hat{l}_0^2 \left(\mathbb{P}_{0,z} [A_{l,l_0}^{0,\epsilon}] + \frac{3}{2} \mathbb{E}_{0,z} [\hat{K} \mathbb{I}_{A_{l,l_0}^{0,\epsilon}}] \right). \end{aligned}$$

Due to the definition of \hat{l}_0 and \hat{K} , we have $D_l^\epsilon(s) + K_l(s) \leq (\hat{l}_0 + \hat{K}\hat{l}_0) + \hat{K}$. Applying this inequality and afterwards Markov's inequality yields

$$\mathbb{P}_{0,z} [A_{l,l_0}^{0,\epsilon}] \leq \mathbb{P}_{0,z} \left[\left[(\hat{l}_0 + \hat{K}\hat{l}_0) + \hat{K} \geq l_0 \right] \right] \leq \frac{\hat{l}_0 + (1 + \hat{l}_0)\mathbb{E}_{0,z} [\hat{K}]}{l_0}.$$

Thanks to *Assumption HomogDiff*, $\mathbb{E}_{0,z}[\hat{K}]$ is uniformly bounded for $z \in \mathbb{G}$ and thus the probability under consideration gets uniformly small the higher we choose l_0 . Due to Lebesgue's dominated convergence theorem with majorant \hat{K} , the same is true for the expectation of $\hat{K}\mathbb{I}_{A_{l,t_0}^{0,\epsilon}}$. \square

Remark 3.1.2. (Comparison to Budhiraja and Ross (2007)).

Because of our cost structure with unbounded K , our proof of the state space truncation is more involved than in the case of Budhiraja and Ross (2007). We do not need to consider the *control space truncation* and the *state space truncation* separately, since our control is bounded anyway as soon as we truncate the state space. That is it follows from $X_l(s) \in [0, l]$ for all $s \in [t, \tau_t^l \wedge T)$ that $\Delta\Theta_s, \Theta_s \in [0, l]$. Therefore, we do not have to assume the value function to be continuous, which is needed in Budhiraja and Ross (2007) to prove the convergence of the control space truncation.

Kushner and Martins (1991) directly consider a truncated optimization problem and do not analyze the truncation itself.

3.1.2 The Markov chain approximation

Thanks to the state space truncation result, we can fix $l \in \mathbb{R}_{\geq 0}$ and focus on the computation of U_l on $[0, T] \times \mathbb{G}_l$. For ease of notation, we thus drop the subscript l in the sequel and write Z, J, U, τ_t instead of Z_l, J_l, U_l, τ_t^l . Let us introduce a three-dimensional h -grid on G_l and approximate the values of U at the grid points by a function U^h . The idea is to approximate the state dynamics (3.1) by a controlled Markov chain to derive U^h . This subsection describes this procedure in detail and introduces the necessary concepts. Afterwards, we go ahead by preparing the convergence proof of the value functions, which is the main result of this chapter. Finally, we explain and implement the numerical scheme that results from the Markov chain approximation.

Without loss of generality, choose $l \geq 1$. For $h > 0$ such that l is an integer multiple of h , take a rectangular grid

$$\mathbb{L}^h := \{h(i, j, k) \mid i, j, k \in \mathbb{N}\}.$$

Define the state space grid for the Markov chain as $\mathbb{G}_l^h := \mathbb{G}_l \cap \mathbb{L}^h$. To conveniently specify the reflection, we consider the slightly larger state space

$$\mathbb{G}_l^{h+} := \mathbb{G}_{l+h} \cap \mathbb{L}^h.$$

Here, we introduce the reflection and constraint boundaries

$$\begin{aligned} \partial_R^h &:= \{z \in \mathbb{G}_l^{h+} \mid \delta = l + h \text{ or } \kappa = l + h\}, \\ \partial^h &:= \{z \in \mathbb{G}_l^{h+} \mid x = 0\}, \\ \partial^0 &:= \{z \in \mathbb{G}_l \mid x = 0\}. \end{aligned}$$

In the following, we will specify transition probabilities to get a controlled Markov chain

$$(Z_n^h)_{n \in \mathbb{N}} \text{ with values in } \mathbb{G}_l^{h+}.$$

Let $(I_n^h)_{n \in \mathbb{N}}$, $I_n^h \in \{0, 1, 2\}$ be the sequence of control actions. Zero corresponds to waiting (diffusion step), one corresponds to trading (control step) and two to a reflection step (see below). For a given control I^h and $E \subset \mathbb{G}_l^{h+}$, we define the probabilistic dynamics of Z^h via

$$Z_0^h := z^h \in \mathbb{G}_l^{h+}, \quad \mathbb{P}[Z_{n+1}^h \in E | \mathcal{F}_n^h] := p^h(Z_n^h, I_n^h, E),$$

where $\mathcal{F}_n^h := \sigma(Z_0^h, \dots, Z_n^h, I_0^h, \dots, I_n^h)$ and

$$p^h(z, i, z) = 1 \text{ for } z \in \partial^h \text{ and } p^h(z, i, z') := q_{(i)}^h(z, z') \text{ for } z \notin \partial^h$$

with transition probabilities $q_{(0)}^h, q_{(1)}^h, q_{(2)}^h : \mathbb{G}_l^{h+} \setminus \partial^h \rightarrow \mathbb{G}_l^{h+}$ to be defined below.

Definition 3.1.3. (Admissible control sequence).

A control sequence I^h is called admissible for the initial value $z^h \in \mathbb{G}_l^{h+}$ if I_n^h is $\sigma(Z_0^h, \dots, Z_n^h, I_0^h, \dots, I_{n-1}^h)$ measurable and for all $n \in \mathbb{N}$

$$\mathbb{P}[I_n^h = 2 | Z_n^h \in \mathbb{G}_l^h] = 0, \quad \mathbb{P}[I_n^h = 2 | Z_n^h \in \partial_R^h \setminus \partial^h] = 1.$$

The set of all these admissible controls is called $\mathcal{A}^h(z^h)$.

Using the concept of admissible controls, we can introduce $U^h : [0, T] \times \mathbb{G}_l^h \rightarrow \mathbb{R}$ by

$$\begin{aligned} U^h(t, z^h) &:= \inf_{I^h \in \mathcal{A}^h(z^h)} J^h(t, z^h, I^h), \\ J^h(t, z^h, I^h) &:= \mathbb{E} \left[\sum_{n=0}^{\eta_t^h - 1} \left(D_n^h + \frac{K_n^h}{2} \frac{h}{K_n^h \vee 1} \right) \frac{h}{K_n^h \vee 1} \mathbb{I}_{\{I_n^h=1\}} \right. \\ &\quad \left. + \left(D_{\eta_t^h}^h + \frac{K_{\eta_t^h}^h}{2} X_{\eta_t^h}^h \right) X_{\eta_t^h}^h \right], \\ \eta_t^h &:= \inf \{ n \in \mathbb{N} | Z_n^h \in \partial^h \text{ or } t_n^h \geq T - t \}, \\ \tau_t^h &:= t + t_{\eta_t^h}^h. \end{aligned} \tag{3.7}$$

The time instances $(t_n^h)_{n \in \mathbb{N}}$ depend on the control

$$t_0^h := 0, \quad t_n^h := \sum_{j=0}^{n-1} \Delta_j^h, \quad \Delta_n^h := \Delta^h(Z_n^h, I_n^h) := \tilde{\Delta}^h(Z_n^h) \mathbb{I}_{\{I_n^h=0\}}.$$

We specify the diffusion time interval $\tilde{\Delta}^h(z)$ below. Intuitively speaking, n counts the number of control actions, but time only elapses in case of a diffusion step. For n corresponding to a control or reflection step, $t_{n+1}^h = t_n^h$. Let us agree on the convention that without loss of generality $I_n^h \neq 1$ for $n \geq \eta_t^h$, i.e. there is no control after τ_t^h .

Let us next state the transition probabilities $q_{(i)}^h$. They are closely related to the difference quotients appearing in a finite difference scheme for the HJB variational inequality from Section 3.2.2. Notice though that the convergence proof for the Markov chain approach does not use the HJB equation directly, but only takes it as an orientation how to sensibly choose the transition probabilities.

Diffusion step: Choice of $q_{(0)}^h$

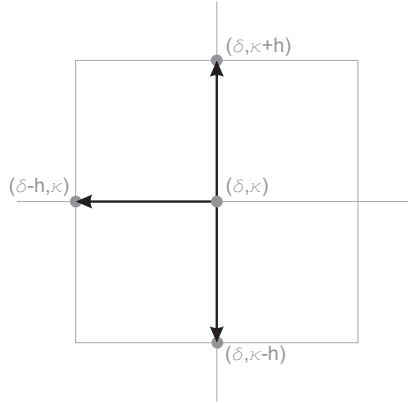


Figure 3.2: Diffusion directions.

In a diffusion step, D decreases due to resilience and K diffuses according to its SDE. For all $z = (\delta, x, \kappa) \in \mathbb{G}_i^h \setminus \partial^h$, define the diffusion transition probabilities

$$\begin{aligned} q_{(0)}^h(z, (\delta-h, x, \kappa)) &:= \rho\delta \frac{\tilde{\Delta}^h(z)}{h} \geq 0, \\ q_{(0)}^h(z, (\delta, x, \kappa-h)) &:= \frac{\sigma^2(\kappa)}{2} \frac{\tilde{\Delta}^h(z)}{h^2} + \mu(\kappa)^- \frac{\tilde{\Delta}^h(z)}{h} \geq 0, \\ q_{(0)}^h(z, (\delta, x, \kappa+h)) &:= \frac{\sigma^2(\kappa)}{2} \frac{\tilde{\Delta}^h(z)}{h^2} + \mu(\kappa)^+ \frac{\tilde{\Delta}^h(z)}{h} \geq 0, \\ q_{(0)}^h(z, z) &:= 1 - \rho\delta \frac{\tilde{\Delta}^h(z)}{h} - \sigma^2(\kappa) \frac{\tilde{\Delta}^h(z)}{h^2} - |\mu(\kappa)| \frac{\tilde{\Delta}^h(z)}{h}. \end{aligned}$$

For all other $z' \in \mathbb{L}^h$, set $q_{(0)}^h(z, z') := 0$. For a given state z , Figure 3.2 outlines the neighboring points with positive transition probability. The self-transition probability $q_{(0)}^h(z, z)$ should be non-negative, which then implies $q_{(0)}^h(z, z') \leq 1$. Therefore, we need to choose the diffusion time step $\tilde{\Delta}^h(z) > 0$ small enough compared to the space step. More precisely, we take

$$\tilde{\Delta}^h(z) \leq h \left(\rho\delta + \frac{\sigma^2(\kappa)}{h} + |\mu(\kappa)| \right)^{-1} = \frac{h^2}{\rho\delta h + \sigma^2(\kappa) + |\mu(\kappa)| h}. \quad (3.8)$$

In particular, $\lim_{h \rightarrow 0} \tilde{\Delta}^h(z) = 0$ for all z . Due to the local boundedness of μ and σ

under *Assumption HomogDiff*, we can find a constant dt^h independent of z such that

$$\tilde{\Delta}^h(z) := dt^h \leq h \left(\rho\delta + \frac{\sigma^2(\kappa)}{h} + |\mu(\kappa)| \right)^{-1}.$$

Remark 3.1.4. (Choice of the time step).

For the time-homogeneous GBM, a feasible choice of $\tilde{\Delta}^h(z)$ is

$$\tilde{\Delta}^h(z) \equiv h \left(\rho l + \frac{\bar{\sigma}^2 l^2}{h} + \bar{\mu} l \right)^{-1} =: dt_{GBM}^h.$$

For the CIR process, we analogously get

$$\tilde{\Delta}^h(z) \equiv \frac{h}{l} \left(\rho + \frac{\bar{\sigma}^2}{h} + \bar{\mu} \right)^{-1} =: dt_{CIR}^h. \quad (3.9)$$

For later use, we show that the diffusion step of the Markov chain is consistent with the continuous state dynamics (3.1). The corresponding Lemma 3.1.5 is the result of a straightforward calculation. It deals with the expectation and variance of $Z_{n+1}^h - Z_n^h$ when $I_n^h = 0$. Define the expectation and variance terms

$$\begin{aligned} m_0(z) &:= \sum_{z' \in \mathbb{L}^h} (z' - z) q_{(0)}^h(z, z'), \\ \sigma_0(z) &:= \sum_{z' \in \mathbb{L}^h} (z' - z - m_0(z))' (z' - z - m_0(z)) q_{(0)}^h(z, z'). \end{aligned}$$

Lemma 3.1.5. (Diffusion consistency conditions).

We get that

$$\begin{aligned} m_0(z) &= \tilde{\Delta}^h(z) (-\rho\delta, 0, \mu(\kappa)), \\ \sigma_0(z) &= \tilde{\Delta}^h(z) \begin{pmatrix} \rho\delta \left(h - \rho\delta \tilde{\Delta}^h(z) \right) & 0 & \rho\delta \tilde{\Delta}^h(z) \mu(\kappa) \\ 0 & 0 & 0 \\ \rho\delta \tilde{\Delta}^h(z) \mu(\kappa) & 0 & \sigma^2(k) - \mu^2(\kappa) \tilde{\Delta}^h(z) \end{pmatrix} \\ &= \tilde{\Delta}^h(z) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma^2(k) \end{pmatrix} + \tilde{\Delta}^h(z) O(h^p) \text{ for some } p > 0. \end{aligned}$$

□

Singular control step: Choice of $q_{(1)}^h$

In a singular control step, we follow the vector $(\kappa, -1, 0)$ times the number of shares that we trade. That is D increases and X decreases. In case of our Markov chain approximation, we buy a small amount of shares which is $O(h)$. As illustrated in Figure 3.3, the neighboring grid points might not lie along the vector $(\kappa, -1, 0)$. Therefore,

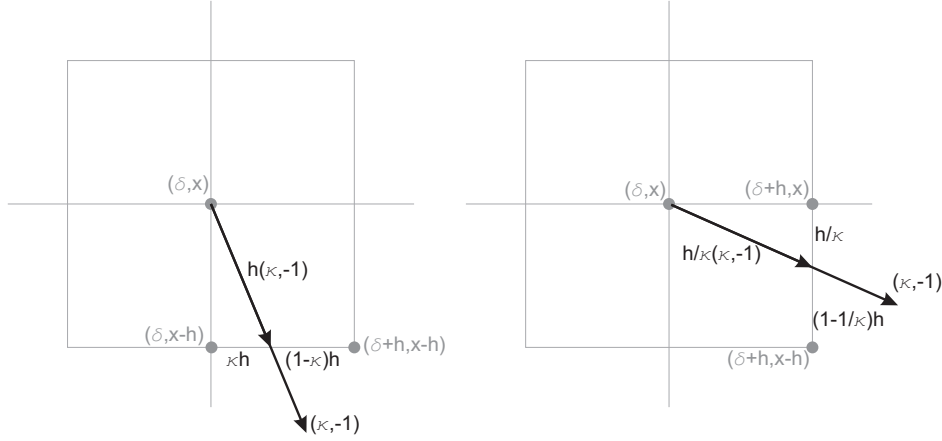


Figure 3.3: Control directions for $\kappa < 1$ (left) and $\kappa \geq 1$ (right).

we choose $q_{(1)}^h$ such that we follow this vector at least on average. This is verified in Lemma 3.1.6. For all $z = (\delta, x, \kappa) \in \mathbb{G}_t^h \setminus \partial^h$, we thus define the control transition probabilities

$$\begin{aligned} q_{(1)}^h(z, (\delta, x - h, \kappa)) &:= (1 - \kappa) \vee 0 \in [0, 1], \\ q_{(1)}^h(z, (\delta + h, x - h, \kappa)) &:= \kappa \wedge \frac{1}{\kappa} \in [0, 1], \\ q_{(1)}^h(z, (\delta + h, x, \kappa)) &:= 0 \vee \left(1 - \frac{1}{\kappa}\right) \in [0, 1] \\ q_{(1)}^h(z, z') &:= 0 \text{ for all other } z' \in \mathbb{L}^h. \end{aligned}$$

In contrast to our specification, the control transition probabilities in Kushner and Martins (1991) and Budhiraja and Ross (2007) are constants.

Define the expectation and variance terms of $Z_{n+1}^h - Z_n^h$ resulting from a control step

$$\begin{aligned} m_1(z) &:= \sum_{z' \in \mathbb{L}^h} (z' - z) q_{(1)}^h(z, z'), \\ \sigma_1(z) &:= \sum_{z' \in \mathbb{L}^h} (z' - z - m_1(z))' (z' - z - m_1(z)) q_{(1)}^h(z, z'). \end{aligned}$$

Lemma 3.1.6. (Control consistency conditions).

We get that

$$\begin{aligned} m_1(z) &= h \left\{ \begin{array}{ll} (\kappa, -1, 0) & \text{if } \kappa \leq 1 \\ (1, -\frac{1}{\kappa}, 0) & \text{otherwise} \end{array} \right\} = \frac{h}{\kappa \vee 1} (\kappa, -1, 0), \\ \sigma_1(z) &= h^2 \left\{ \begin{array}{ll} \left(\begin{array}{ccc} \kappa(1 - \kappa) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) & \text{if } \kappa \leq 1 \\ \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & \frac{\kappa - 1}{\kappa^2} & 0 \\ 0 & 0 & 0 \end{array} \right) & \text{otherwise} \end{array} \right\} = O(h^2). \end{aligned}$$

□

Normal reflection step: Choice of $q_{(2)}^h$

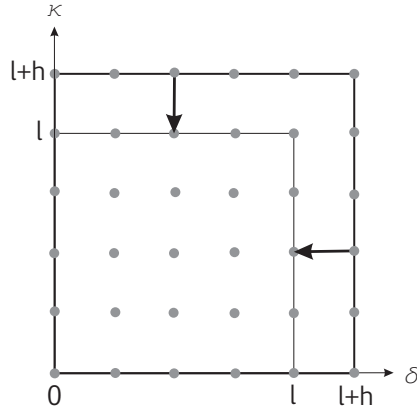


Figure 3.4: Reflection directions.

Starting in \mathbb{G}_l^h , the only possibilities that our Markov chain reaches ∂_R^h is by diffusion when K gets larger than l or by control when D gets larger than l . For $z \in \partial_R^h$, we do a normal reflection as illustrated in Figure 3.4:

$$\begin{aligned} q_{(2)}^h((\delta, x, l+h), (\delta, x, l)) &:= 1, \\ q_{(2)}^h((l+h, x, \kappa), (l, x, \kappa)) &:= 1, \\ q_{(2)}^h(z, z') &:= 0 \text{ for all other } z, z' \in \mathbb{L}^h. \end{aligned}$$

3.1.3 Continuous time interpolation

For notational convenience, we set without loss of generality $t = 0$. Some more notation is needed in order to translate the discrete time, discrete state Markov decision problem from the preceding subsection into a continuous time formulation, where the approximating processes are constant on each of the intervals $(t_n^h, t_{n+1}^h]$.

The (\mathcal{F}_n^h) stopping time

$$n^h(s) := \max\{n \in \mathbb{N} | t_n^h < s\}$$

is key for the continuous time interpolation. Set $\mathcal{F}^h(s) := \mathcal{F}_{n^h(s)}^h$ for $s \in [0, T]$. In the same manner, we can interpret $I^h(s) := I_{n^h(s)}^h$ and $Z^h(s) := Z_{n^h(s)}^h$ with $Z^h = (D^h, X^h, K^h)$ as continuous time processes. Set $\Delta Z_n^h := Z_{n+1}^h - Z_n^h$ and define the controlled Markov chain's

- drift part of the diffusion

$$B_n^h := \sum_{k=0}^{n-1} \mathbb{E} [\Delta Z_k^h | \mathcal{F}_k^h] \mathbb{I}_{\{I_k^h=0\}},$$

- martingale part of the diffusion

$$S_n^h := \sum_{k=0}^{n-1} (\Delta Z_k^h - \mathbb{E} [\Delta Z_k^h | \mathcal{F}_k^h]) \mathbb{I}_{\{I_k^h=0\}}, \quad (3.10)$$

- proxy for the strategy

$$\Theta_n^h := \sum_{k=0}^{n-1} \frac{h}{K_k^h \vee 1} \mathbb{I}_{\{I_k^h=1\}},$$

- martingale part corresponding to singular control

$$E_n^h := \sum_{k=0}^{n-1} \left(\Delta Z_k^h - \frac{h}{K_k^h \vee 1} (K_k^h, -1, 0) \right) \mathbb{I}_{\{I_k^h=1\}}$$

- and reflection part

$$R_n^h := - \sum_{k=0}^{n-1} \Delta Z_k^h \mathbb{I}_{\{I_k^h=2\}}.$$

The corresponding continuous time processes start at zero and are left-continuous step functions defined as

$$B^h(s) := B_{n^h(s)}^h, S^h(s) := S_{n^h(s)}^h, \Theta^h(s) := \Theta_{n^h(s)}^h, E^h(s) := E_{n^h(s)}^h, R^h(s) := R_{n^h(s)}^h.$$

The above interpretation of these processes is made precise by the next lemma.

Lemma 3.1.7. *We get that*

$$\begin{aligned} Z^h(s) &= z^h + B^h(s) + S^h(s) + \int_{[0,s)} (K^h(r), -1, 0) d\Theta^h(r) + E^h(s) - R^h(s), \\ B^h(s) &= \int_{[0,s)} (-\rho D^h(r), 0, \mu(K^h(r))) dr \quad \text{for } s \in [0, \infty). \end{aligned}$$

3.1.4 Time rescaling

For the convergence result, we need weakly convergent subsequences of the processes introduced in the last section when $h \rightarrow 0$. Thanks to Prokhorov's Theorem, this can be obtained by showing tightness. But due to the singular control nature of our problem, it is difficult to get this tightness directly. Therefore, we need the idea by Kushner to *stretch out time*: In each control instance, we artificially expand the time by the size of the trade and thus transform singular to classical control processes. In the next subsection, we can then prove tightness for the stretched out processes using the Aldous-Kurtz criterion. This time rescaling is later being undone to obtain the convergence result.

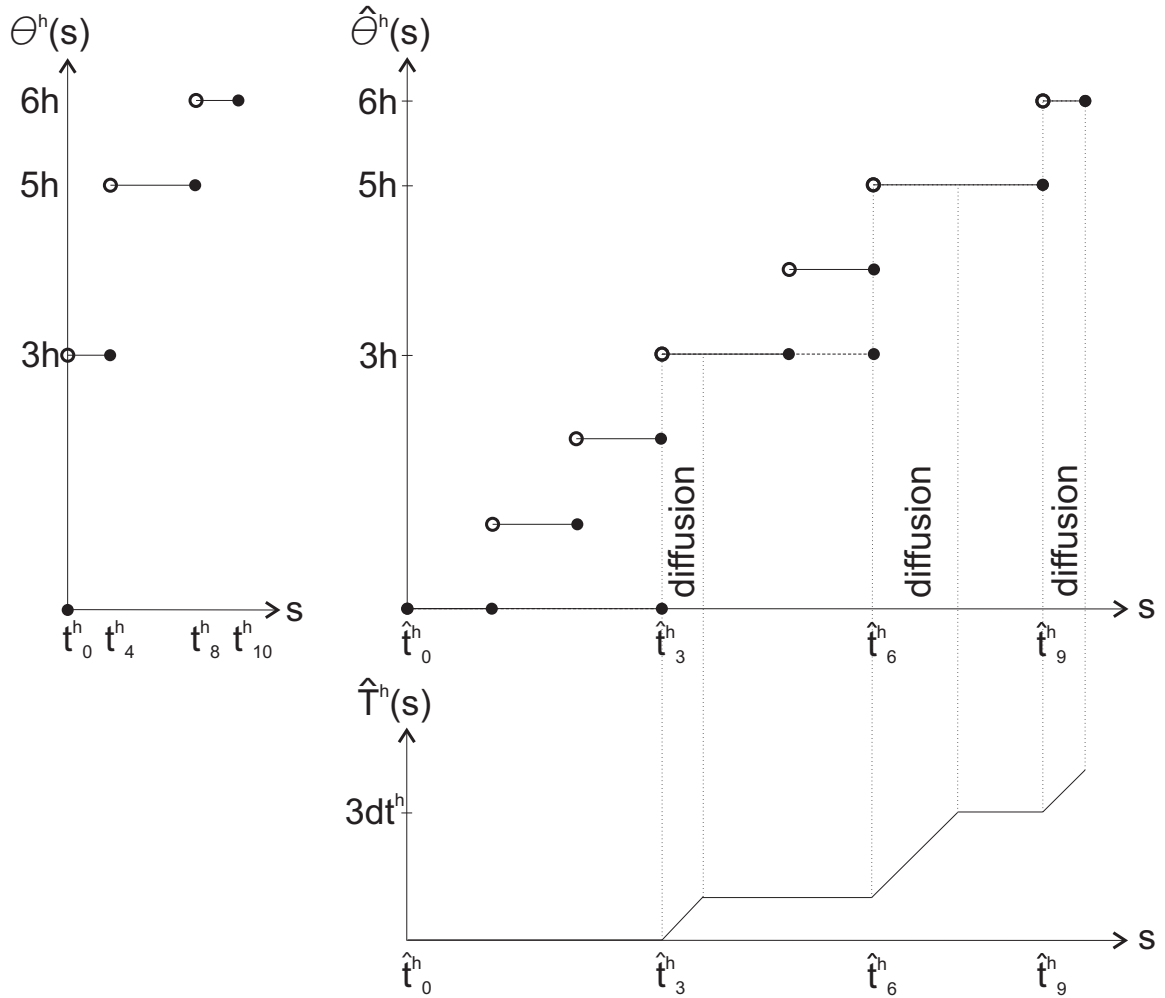


Figure 3.5: This example illustrates the continuous time interpolation and time rescaling. For simplicity, assume $K_n \leq 1$ for all n . We have a given deterministic control sequence $(I_n^h)_{n=0,\dots,9} = (1, 1, 1, 0, 1, 1, 0, 0, 1, 0)$. Then $(\Theta_n^h)_{n=0,\dots,10} = h(0, 1, 2, 3, 3, 4, 5, 5, 6, 6)$ and $(t_n^h)_{n=0,\dots,10} = dt^h(0, 0, 0, 0, 1, 1, 1, 2, 3, 3, 4)$. The plot of $\hat{\Theta}^h$ also contains $\Theta^h \circ \hat{T}^h$ as a dashed line. They differ on $(\hat{t}_1^h, \hat{t}_3^h] \cup (\hat{t}_5^h, \hat{t}_6^h]$, i.e. when more than one control step is done in a row.

Definition 3.1.8. (Rescaled time process).

Define the rescaled time grid $\hat{t}_0^h := 0, \hat{t}_n^h := \sum_{j=0}^{n-1} \hat{\Delta}_j^h$ with time intervals

$$\hat{\Delta}_n^h := \left\{ \begin{array}{ll} \Delta_n^h & \text{for } I_n^h = 0 \\ \frac{h}{K_n^h \vee 1} & \text{for } I_n^h = 1 \\ 0 & \text{for } I_n^h = 2 \end{array} \right\}.$$

The rescaled time process \hat{T}^h is the unique continuous, nondecreasing process that satisfies $\hat{T}^h(0) = 0$ and for $s \in (\hat{t}_n^h, \hat{t}_{n+1}^h)$

$$\frac{\partial}{\partial s} \hat{T}^h(s) = \left\{ \begin{array}{ll} 1 & \text{for } I_n^h = 0 \\ 0 & \text{for } I_n^h = 1, 2 \end{array} \right\}.$$

From this definition, it follows that $\hat{T}^h(\hat{t}_n^h) = t_n^h$ and $\hat{T}^h(\hat{t}_{n+1}^h) - \hat{T}^h(\hat{t}_n^h) = \Delta_n^h$. The (\mathcal{F}_n^h) -stopping time $\hat{n}^h(s)$ is defined as

$$\hat{n}^h(s) := \max \{n \in \mathbb{N} | \hat{t}_n^h < s\} \leq 2 \left(\left\lceil \frac{s}{\min_{z \in \mathbb{G}_t^h} \tilde{\Delta}^h(z)} \right\rceil + \left\lceil \frac{s}{\frac{h}{l}} \right\rceil \right), \quad (3.11)$$

where the multiplier two is due to reflection, the first summand counts the maximum number of diffusion steps and the second summand the maximum number of control steps. Let $\hat{\mathcal{F}}^h(s) := \mathcal{F}_{\hat{n}^h(s)}^h$ and introduce the rescaled process

$$\hat{I}^h(s) := I_{\hat{n}^h(s)}^h.$$

Define $\hat{Z}^h, \hat{B}^h, \hat{S}^h, \hat{\Theta}^h, \hat{E}^h, \hat{R}^h$ analogously.

Remark 3.1.9. Notice that \hat{n}^h and $n^h \circ \hat{T}^h$ are different. It is a slight error in Budhiraja and Ross (2007) that these processes are taken to be equal. Thus, $\hat{I}^h \neq I^h \circ \hat{T}^h$. This difference is illustrated for $\hat{\Theta}^h$ and $\Theta^h \circ \hat{T}^h$ in Figure 3.5.

The following lemma contains

$$\hat{\tau}_0^h := \inf \left\{ s \in [0, \infty) | \hat{Z}^h(s) \in \partial^h \text{ or } \hat{T}^h(s) \geq T \right\}.$$

Lemma 3.1.10. *We get that*

$$\begin{aligned} \hat{Z}^h(s) &= z^h + \hat{B}^h(s) + \hat{S}^h(s) + \int_{[0,s)} \left(\hat{K}^h(r), -1, 0 \right) d\hat{\Theta}^h(r) + \hat{E}^h(s) - \hat{R}^h(s), \\ \hat{B}^h(s) &= \int_{[0,s)} \left(-\rho \hat{D}^h(r), 0, \mu \left(\hat{K}^h(r) \right) \right) d\hat{T}^h(r) \quad \text{for } s \in [0, \infty) \text{ and} \\ J^h(0, z^h, I^h) &= \mathbb{E} \left[\int_{[0, \hat{\tau}_0^h)} \left(\hat{D}^h(s) + \frac{\hat{K}^h(s)}{2} \Delta \hat{\Theta}^h(s) \right) d\hat{\Theta}^h(s) \right. \\ &\quad \left. + \left(\hat{D}^h(\hat{\tau}_0^h) + \frac{\hat{K}^h(\hat{\tau}_0^h)}{2} \hat{X}^h(\hat{\tau}_0^h) \right) \hat{X}^h(\hat{\tau}_0^h) \right] \end{aligned}$$

Proof. The first two equations follow analogously to Lemma 3.1.7. In the expectation of the third equation, plug in the definition of the time rescaled processes and $\hat{n}^h(\hat{\tau}_0^h) = \eta_0^h$

$$\begin{aligned} & \mathbb{E} \left[\int_{[0, \hat{\tau}_0^h]} \left(D_{\hat{n}^h(s)}^h + \frac{K_{\hat{n}^h(s)}^h}{2} \Delta \Theta_{\hat{n}^h(s)}^h \right) d\Theta_{\hat{n}^h(s)}^h + \left(D_{\eta_0^h}^h + \frac{K_{\eta_0^h}^h}{2} X_{\eta_0^h}^h \right) X_{\eta_0^h}^h \right] \\ &= \mathbb{E} \left[\sum_{n=0}^{\eta_0^h-1} \left(D_n^h + \frac{K_n^h}{2} \frac{h}{K_n^h \vee 1} \right) \frac{h}{K_n^h \vee 1} \mathbb{1}_{\{I_n^h=1\}} + \left(D_{\eta_0^h}^h + \frac{K_{\eta_0^h}^h}{2} X_{\eta_0^h}^h \right) X_{\eta_0^h}^h \right]. \end{aligned}$$

This term equals the definition of $J^h(0, z^h, I^h)$ from (3.7). \square

3.1.5 Convergence and tightness of the rescaled processes

First of all, we show that the process \hat{E}^h converges to zero in probability as $h \rightarrow 0$. Therefore, we do not need to consider it in the following tightness result Lemma 3.1.12. The idea of the proofs is again taken from Budhiraja and Ross (2007).

Lemma 3.1.11. *As $h \rightarrow 0$, \hat{E}^h converges to zero in probability in $\mathcal{D}([0, \infty) : \mathbb{R}^3)$, which is the space of càglàd functions endowed with the Skorokhod topology.*

Proof. Due to the control consistency condition in Lemma 3.1.6, $(E_n^h)_{n \in \mathbb{N}}$ must be a discrete (\mathcal{F}_n^h) -martingale. Since $\hat{n}^h(s)$ from (3.11) is bounded, we can apply Doob's *Optional-Sampling Theorem* in order to get for $0 \leq r \leq s$ that

$$\mathbb{E} \left[\hat{E}^h(s) \middle| \hat{\mathcal{F}}^h(r) \right] = \mathbb{E} \left[E_{\hat{n}^h(s)}^h \middle| \mathcal{F}_{\hat{n}^h(r)}^h \right] = E_{\hat{n}^h(r)}^h = \hat{E}^h(r).$$

That is the *Burkholder-Davis-Gundy inequality* can be applied to (\hat{E}_s^h) :

$$\mathbb{E} \left[\sup_{r \in [0, s]} \left| \hat{E}^h(r) \right| \right]^2 \leq \text{const} \mathbb{E} \left[\left\langle \hat{E}^h \right\rangle_s^{\frac{1}{2}} \right]^2 \leq \text{const} \mathbb{E} \left[\left\langle \hat{E}^h \right\rangle_s \right] \leq \left[\frac{sl}{h} \right] O(h^2) \xrightarrow{h \rightarrow 0} 0. \quad (3.12)$$

We also used Jensen's inequality and control consistency from Lemma 3.1.6. \square

Lemma 3.1.12. (Tightness).

The laws of $\left\{ \left(\hat{H}^h, \hat{\tau}_0^h \right), h > 0 \right\}$ with $\hat{H}^h := \left(\hat{Z}^h, \hat{T}^h, \hat{\Theta}^h, \hat{R}^h, \hat{B}^h, \hat{S}^h \right)$ form a tight family of distributions on

$$\mathcal{D}([0, \infty) : \mathbb{R}^{14}) \times [0, \infty].$$

Proof. In order to prove tightness of a family of distributions belonging to some processes, we use the Aldous-Kurtz criterion as in Kurtz (1981), Theorem 2.7 b. It says that for each component ϕ^h of \hat{H}^h , we need to check that for all $M > 0$

$$\lim_{s \rightarrow 0} \limsup_{h \rightarrow 0} \sup_{\tau \leq M} \mathbb{E} \left[1 \wedge \left| \phi^h(\tau + s) - \phi^h(\tau) \right| \right] = 0.$$

Thus $\{\hat{T}^h\}$ is tight, since $\hat{T}^h(\tau + s) - \hat{T}^h(\tau) \leq s$. Due to our stretching out of the time,

$$\hat{\Theta}^h(\tau + s) - \hat{\Theta}^h(\tau) \leq s + h. \quad (3.13)$$

This yields the tightness for the control process. The tightness of $\{\hat{B}^h\}$ is also immediate with Lemma 3.1.10, $\hat{D}^h, \hat{K}^h, \mu(\cdot)$ being bounded and \hat{T}^h being Lipschitz continuous. We still have to deal with $\{\hat{S}^h\}$. Similar to the proof of Lemma 3.1.11, one can show that \hat{S}^h is a martingale and similar to (3.12)

$$\left(\mathbb{E} \left[\left| \hat{S}^h(\tau + s) - \hat{S}^h(\tau) \right| \right] \right)^2 \leq \text{const} \mathbb{E} \left[\text{Trace} \left(\left\langle \hat{S}^h \right\rangle_{\tau+s} - \left\langle \hat{S}^h \right\rangle_{\tau} \right) \right]. \quad (3.14)$$

A detailed calculation in the proof of Lemma 3.1.13 below shows

$$\begin{aligned} \left\langle \hat{S}^h \right\rangle_s &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sum_{k=0}^{\hat{n}^h(s)-1} \sigma^2 (K_k^h) \tilde{\Delta}^h (Z_k^h) \mathbb{I}_{\{I_k^h=0\}} \\ &+ \sum_{k=0}^{\hat{n}^h(s)-1} \tilde{\Delta}^h (Z_k^h) O(h^p) \mathbb{I}_{\{I_k^h=0\}} \end{aligned} \quad (3.15)$$

for the same $p > 0$ as in Lemma 3.1.5. We see from (3.14) and (3.15) that the Aldous-Kurtz criterion is satisfied, since

$$\sum_{k=\hat{n}^h(\tau)}^{\hat{n}^h(\tau+s)-1} \tilde{\Delta}^h (Z_k^h) \mathbb{I}_{\{I_k^h=0\}} \leq \left| \hat{T}^h(\tau + s) - \hat{T}^h(\tau) \right| \leq s.$$

It only remains to prove the tightness for the state and reflection processes. Define

$$\hat{\underline{Z}}^h(s) := z^h + \hat{B}^h(s) + \hat{S}^h(s) + \int_{[0,s)} \left(\hat{K}^h(r), -1, 0 \right) d\hat{\Theta}^h(r) + \hat{E}^h(s). \quad (3.16)$$

Then $\hat{Z}^h = \hat{\underline{Z}}^h - \hat{R}^h$ thanks to Lemma 3.1.10 and the distribution of $\hat{\underline{Z}}^h$ is tight as we argued above. Moreover, \hat{Z}^h can be expressed as image of $\hat{\underline{Z}}^h$ under the Skorokhod map. Since the Skorokhod map is Lipschitz continuous, \hat{Z}^h satisfies the Aldous-Kurtz criterion and, as a consequence, the reflection process must be tight, too.

The tightness of $\{\hat{\tau}_0^h\}$ follows from $[0, \infty]$ being compact. \square

3.1.6 Properties of the rescaled limit processes

For $z^0 \in \mathbb{G}_l$, let (z^h) be a sequence with $Z_0^h = z^h \in \mathbb{G}_l^+$ and $\lim_{h \rightarrow 0} z^h = z^0$. Thanks to the tightness in Lemma 3.1.12, there exists $\hat{H} = (\hat{Z}, \hat{T}, \hat{\Theta}, \hat{R}, \hat{B}, \hat{S}), \hat{\tau}_0$ and a subsequence such that the law of $(\hat{H}^h, \hat{\tau}_0^h)$ converges weakly to the law of $(\hat{H}, \hat{\tau}_0)$ as $h \rightarrow 0$:

$$\left(\hat{H}^h, \hat{\tau}_0^h \right) \xrightarrow{w} \left(\hat{H}, \hat{\tau}_0 \right). \quad (3.17)$$

For notational convenience, we still take h to index this weakly convergent subsequence. Due to the *Skorokhod representation*, we can without loss of generality assume that all processes are defined on the same probability space and that we have convergence with probability one in the space of càglàd functions instead of weak convergence of the laws. Slightly modifying the corresponding proof in Budhiraja and Ross (2007), we get that the properties of the elements of the subsequence are preserved in the limit:

Lemma 3.1.13. *The limit $(\hat{H}, \hat{\tau}_0)$ satisfies:*

1. \hat{T} is nondecreasing and Lipschitz continuous with coefficient one.
2. $\hat{B}(s) = \int_{[0,s)} \left(-\rho \hat{D}(r), 0, \mu(\hat{K}(r)) \right) d\hat{T}(r)$, $s \geq 0$.
3. $\hat{S}_1 = \hat{S}_2 \equiv 0$ a.s. and the third component \hat{S}_3 is a continuous martingale for the filtration $\hat{\mathcal{F}}(s) := \sigma(\hat{H}(r), r \in [0, s])$. Its quadratic variation is given by

$$\langle \hat{S}_3 \rangle (s) = \int_{[0,s)} \sigma^2 \left(\hat{K}(r) \right) d\hat{T}(r), \quad s \geq 0. \quad (3.18)$$

4. $\hat{\Theta}$ is nondecreasing and continuous.
5. \hat{R} is componentwise nondecreasing and continuous with

$$\int_{[0,\infty)} \mathbb{I}_{\{\hat{D}(r+) < l\}} d\hat{R}_1(r) = \int_{[0,\infty)} \mathbb{I}_{\{\hat{K}(r+) < l\}} d\hat{R}_2(r) = 0. \quad (3.19)$$

6. \hat{Z} is continuous, takes values in \mathbb{G}_l a.s. and

$$\hat{Z}(s) = z^0 + \hat{B}(s) + \hat{S}(s) + \int_{[0,s)} \left(\hat{K}(r), -1, 0 \right) d\hat{\Theta}(r) - \hat{R}(s), \quad s \geq 0. \quad (3.20)$$

Proof. We consider each of the statements separately.

1. The first statement follows directly from the corresponding properties of \hat{T}^h .
2. Consider

$$\begin{aligned} & \left| \hat{B}^h(s) - \int_{[0,s)} \left(-\rho \hat{D}(r), 0, \mu(\hat{K}(r)) \right) d\hat{T}(r) \right| \\ & \leq \int_{[0,s)} \left| \left(-\rho \hat{D}^h(r), 0, \mu(\hat{K}^h(r)) \right) - \left(-\rho \hat{D}(r), 0, \mu(\hat{K}(r)) \right) \right| d\hat{T}(r) \\ & + \int_{[0,s)} |d\hat{T}^h(r) - d\hat{T}(r)| \sup_{r \in [0,s)} \left| \left(-\rho(\hat{D}^h(r) - \hat{D}(r)), 0, \mu(\hat{K}^h(r)) - \mu(\hat{K}(r)) \right) \right| \\ & + \left| \int_{[0,s)} \left(-\rho \hat{D}(r), 0, \mu(\hat{K}(r)) \right) d\hat{T}^h(r) - \int_{[0,s)} \left(-\rho \hat{D}(r), 0, \mu(\hat{K}(r)) \right) d\hat{T}(r) \right| \end{aligned} \quad (3.21)$$

for $h \rightarrow 0$. The integrand of the first integral of the right-hand side is bounded and so we can use Lebesgue's dominated convergence theorem recalling that μ is assumed to be continuous. The total variation in the second summand is bounded by $2s$. The convergence with respect to the Skorokhod topology corresponds to uniform convergence when the limit is continuous. As we see below, \hat{D} and \hat{K} are continuous. Therefore, the supremum in the second summand converges to zero. The integrands in the last summand are identical to each other, continuous and do not depend on h . Hence, we can use the definition of weak convergence for measures to conclude that the last summand converges to zero.

Therefore, the entire term (3.21) converges to zero with probability one as $h \rightarrow 0$.

3. This is the main part of this proof. Due to the definition of the diffusion step,

$$\sup_{r \in [0, s]} \left| \hat{S}^h(r+) - \hat{S}^h(r) \right| \leq \text{const } h.$$

That is the jumps of \hat{S}^h are $O(h)$ and so \hat{S} must be continuous by Ethier and Kurtz (1986), Theorem 3.10.2. As in the proof of Lemma 3.1.11, we can show that \hat{S}^h is an $\hat{\mathcal{F}}^h$ -martingale. Using the diffusion consistency conditions,

$$\begin{aligned} \langle S^h \rangle_n &= \sum_{k=0}^{n-1} \mathbb{E} \left[(S_{k+1}^h - S_k^h)' (S_{k+1}^h - S_k^h) \middle| \mathcal{F}_k^h \right] \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sum_{k=0}^{n-1} \sigma^2 (K_k^h) \tilde{\Delta}^h (Z_k^h) \mathbb{I}_{\{I_k^h=0\}} \\ &+ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sum_{k=0}^{n-1} \tilde{\Delta}^h (Z_k^h) O(h^p) \mathbb{I}_{\{I_k^h=0\}} \end{aligned}$$

for $p > 0$ from Lemma 3.1.5. For $\langle \hat{S}^h \rangle_s$, we get the same term, but with n being replaced by $\hat{n}^h(s)$ and the second sum being of the order $sO(h^p)$. It follows that $\hat{S}_1 = \hat{S}_2 \equiv 0$ for $h \rightarrow 0$ as desired.

For each fixed $s \in [0, \infty)$, we show in a moment using the *de la Vallée Poussin Theorem* that $\{(\hat{S}_3^h(s))^2, h > 0\}$ is uniformly integrable. Therefore, not only

$$\hat{S}_3^h \text{ and } \left(\hat{S}_3^h \right)^2 - \langle \hat{S}_3^h \rangle,$$

but also

$$\hat{S}_3 \text{ and } \hat{S}_3^2 - \int_{[0, \cdot)} \sigma^2 \left(\hat{K}(r) \right) d\hat{T}(r)$$

must be martingales. Thus, the uniqueness of the quadratic variation yields (3.18).

According to the Burkholder-Davis-Gundy inequality,

$$\mathbb{E} \left[\left| \hat{S}_3^h(s) \right|^4 \right] \leq \text{const} \mathbb{E} \left[\left\langle \hat{S}_3^h \right\rangle^2 (s) \right] \leq \text{const} \left(\sup_{\kappa \in [0, l+h]} \sigma^2(\kappa) + O(h^p) \right)^2 s^2.$$

That is $\sup_{h \in (0,1]} \mathbb{E}[|\hat{S}_3^h(s)|^4] < \infty$, which yields the uniform integrability.

4. We have nondecreasing $\hat{\Theta}^h$ and, due to (3.13), $\hat{\Theta}$ cannot have any jumps.
5. Since \hat{R}^h is nondecreasing, this also holds for its limit. Because of the Lipschitz continuity of the Skorokhod map and the continuity of the limit of (3.16), we get continuity of \hat{R} as well as (3.19).
6. The statement follows from Lemma 3.1.10 and

$$\mathbb{P} \left[\hat{Z}^h(s) \in [0, l+h] \times [0, l] \times [0, l+h] \right] = 1.$$

□

3.1.7 Undo time rescaling

Having introduced the time rescaling method in order to prove tightness and the existence of convergent subsequences, we now translate the resulting limit processes back to the original time scale by using the left-continuous inverse of \hat{T} . In this subsection, we make sure that the properties of the rescaled processes listed in Lemma 3.1.13 are preserved under this change back to the original time scale. A first step in this direction is the following lemma. It shows that the left-continuous inverse of \hat{T} is well-defined. Its proof can mostly be adopted from Budhiraja and Ross (2007).

Lemma 3.1.14. (Limit of rescaled time process is surjective).

$$\lim_{s \rightarrow \infty} \hat{T}(s) = \infty \quad a.s.$$

Proof. Suppose for a contradiction that there exists $\epsilon > 0$, $T_0 > 1$ with

$$\mathbb{P} \left[\sup_{s \geq 0} \hat{T}(s) < T_0 - 1 \right] > \epsilon.$$

Thus, we get a contradiction by showing that there exists $M > 0$ with

$$\mathbb{P} \left[\hat{T}(T_0 + M) < T_0 - 1 \right] = \liminf_{h \rightarrow 0} \mathbb{P} \left[\hat{T}^h(T_0 + M) < T_0 - 1 \right] \leq \frac{\epsilon}{2}. \quad (3.22)$$

According to Lemma 3.1.15 below, we can find $M > 0$ such that for all $h \in (0, 1]$

$$\mathbb{P} \left[\Theta^h(T_0) \geq M \right] \leq \frac{\mathbb{E} \left[\Theta^h(T_0) \right]}{M} \leq \frac{\epsilon}{2}.$$

Hence, we can estimate

$$\begin{aligned} & \mathbb{P} \left[\hat{T}^h(T_0 + M) < T_0 - 1 \right] \\ & \leq \mathbb{P} \left[\left\{ \hat{T}^h(T_0 + M) < T_0 - 1 \right\} \cap \left\{ \Theta^h(T_0) < M \right\} \right] + \mathbb{P} \left[\Theta^h(T_0) \geq M \right] \\ & \leq \mathbb{P} \left[\hat{T}^h(T_0 + \Theta^h(T_0)) < T_0 - 1 \right] + \frac{\epsilon}{2}. \end{aligned}$$

For (3.22), it only remains to convince ourselves that the last probability converges to zero for $h \searrow 0$. Consider

$$T_0 + \Theta^h(T_0) \geq \sum_{k=0}^{n^h(T_0)-1} \left(\Delta_k^h \mathbb{I}_{\{I_k^h=0\}} + \frac{h}{K_k^h \vee 1} \mathbb{I}_{\{I_k^h=1\}} \right) = \hat{t}_{n^h(T_0)}^h.$$

Due to the boundedness of $\mu, \sigma(\cdot)$ on $[0, l]$ and (3.8), we get for small h that

$$\hat{T}^h(T_0 + \Theta^h(T_0)) \geq \hat{T}^h(\hat{t}_{n^h(T_0)}^h) = t_{n^h(T_0)}^h \geq T_0 - \sup_{z \in \mathbb{G}_l^h} \tilde{\Delta}^h(z) \geq T_0 - 1.$$

□

In the proof of Lemma 3.1.14, we needed the expectation of Θ^h to be bounded. We show this in the following lemma. Its proof utilizes the specific features of our optimal execution problem. Notice that the proxy Θ^h for the strategy resulting from a control I^h is not pathwise bounded. This is due to the fact that a control step with $I_k^h = 1$ leads to an increase of Θ^h by at least $\frac{h}{l}$, but the number of shares still to be traded X^h stays constant with possibly positive probability, which is less than or equal to $1 - \frac{1}{l}$.

Lemma 3.1.15. *For all $T_0 \in [0, \infty)$ and admissible controls,*

$$\sup_{h \in (0, 1]} \mathbb{E} [\Theta^h(T_0)] \leq l^2 < \infty.$$

Proof. For

$$n_1^h(X^h) := \inf \{ n \in \mathbb{N} \mid X_n^h = 0 \},$$

we have

$$\mathbb{E}[\Theta^h(T_0)] \leq h \mathbb{E} \left[\sum_{k=0}^{n_1^h(X^h)-1} \mathbb{I}_{\{I_k^h=1\}} \right],$$

where the sum represents the number of control instances until all shares are bought. Define the dynamics of \tilde{X}^h by $\tilde{X}_0^h := l$ and for $E \subset \{0, h, 2h, \dots, l\}$,

$$\mathbb{P} \left[\tilde{X}_{n+1}^h \in E \mid \mathcal{F}_n^h \right] := \tilde{p}^h \left(\tilde{X}_n^h, I_n^h, E \right)$$

with

$$\begin{aligned}\tilde{p}^h(\tilde{x}, 0, \tilde{x}) &= \tilde{p}^h(\tilde{x}, 2, \tilde{x}) = 1, \\ \tilde{p}^h(\tilde{x}, 1, \tilde{x}) &= 1 - \frac{1}{l}, \\ \tilde{p}^h(\tilde{x}, 1, \tilde{x} - h) &= \frac{1}{l}\end{aligned}$$

and for $i \in \{0, 1, 2\}$ and all other \tilde{x}' , set $\tilde{p}^h(\tilde{x}, i, \tilde{x}') = 0$. Then \tilde{X}_n^h stochastically dominates X_n^h for all $n \in \mathbb{N}$. Hence,

$$\mathbb{E} \left[\sum_{k=0}^{n_1^h(X^h)-1} \mathbb{I}_{\{I_k^h=1\}} \right] \leq \mathbb{E} \left[\sum_{k=0}^{n_1^h(\tilde{X}^h)-1} \mathbb{I}_{\{I_k^h=1\}} \right] \leq \mathbb{E} [n_1^h(Y^h)],$$

where Y^h is a classical asymmetric random walk with negative drift defined by $Y_0^h := l$,

$$\mathbb{P} [Y_{n+1}^h - Y_n^h = \tilde{x}] = \begin{cases} 1 - \frac{1}{l} & \text{for } \tilde{x} = 0 \\ \frac{1}{l} & \text{for } \tilde{x} = -h \end{cases}.$$

Then

$$\begin{aligned}\mathbb{E} [n_1^h(Y^h)] &= \sum_{k=\frac{l}{h}}^{\infty} k \mathbb{P} [n_1^h(Y^h) = k] \\ &= \sum_{k=0}^{\infty} \left(\frac{l}{h} + k \right) \left(\frac{1}{l} \right)^{\frac{l}{h}} \left(1 - \frac{1}{l} \right)^k \binom{\frac{l}{h} + k - 1}{k} = \frac{l^2}{h}.\end{aligned}$$

The last equation holds, since induction over $N \in \mathbb{N}$ shows

$$\sum_{k=0}^{\infty} \left(1 - \frac{1}{l} \right)^k \binom{N+k}{k} = l^{N+1}.$$

□

The left-continuous inverse of \hat{T}

$$T(s) := \inf \left\{ r \geq 0 \mid \hat{T}(r) \geq s \right\}$$

is finite a.s. thanks to Lemma 3.1.14. Moreover, T is nondecreasing and

$$\begin{aligned}\hat{T}(s) &\leq s, & T(s) &\geq s, \\ \hat{T}(T(s)) &= s, & T(\hat{T}(s)) &\leq s, \\ \lim_{s \rightarrow \infty} T(s) &= \infty \text{ a.s.}, & \hat{T}(s) \in [0, r] &\Leftrightarrow s \in [0, T(r+)].\end{aligned}$$

Define the time-inverted processes and stopping time by

$$H := \hat{H} \circ T, \quad \tau_0 := \hat{T}(\hat{\tau}_0). \quad (3.23)$$

Due to $\{T(r) \leq s\} = \{\hat{T}(s) \geq r\}$, we know that $T(r)$ is an $\hat{\mathcal{F}}$ -stopping time for each fixed $r \in [0, \infty)$. Therefore, $H(s)$ is $\hat{\mathcal{F}}(T(s)) =: \mathcal{F}^0(s)$ measurable such that

$$\mathcal{F}(s) := \sigma(H(r), r \in [0, s]) \subset \mathcal{F}^0(s).$$

The same properties as in Lemma 3.1.13 also hold for the time inverted processes.

Lemma 3.1.16. *For all $s \in [0, \infty)$, the time inverted limit (H, τ_0) satisfies:*

1. T is nondecreasing and left-continuous.
2. $B(s) = \int_{[0,s]} (-\rho D(r), 0, \mu(K(r))) dr$
3. $S_1 = S_2 \equiv 0$ a.s. and the third component S_3 is a continuous \mathcal{F}^0 -martingale with

$$\langle S_3 \rangle (s) = \int_{[0,s]} \sigma^2(K(r)) dr, \quad (3.24)$$

$$S_3(s) = \int_{[0,s]} \sigma(K(r)) dW(r) \quad (3.25)$$

for a Brownian motion W on an extension of $(\Omega, \mathcal{F}, (\mathcal{F}^0), \mathbb{P})$.

4. Θ is nondecreasing and left-continuous, i.e. $\Theta \in \tilde{\mathcal{A}}_0$ by setting $\Theta_T = \Theta_{T+}$.
5. R is componentwise nondecreasing and left-continuous with

$$\int_{[0,\infty)} \mathbb{I}_{\{D(r+) < l\}} dR_1(r) = \int_{[0,\infty)} \mathbb{I}_{\{K(r+) < l\}} dR_2(r) = 0. \quad (3.26)$$

6. Z is left-continuous, takes values in \mathbb{G}_l a.s. and

$$\begin{aligned} Z(s) &= z^0 + B(s) + S(s) + \int_{[0,s]} (K(r), -1, 0) d\Theta(r) - R(s), \quad (3.27) \\ dD(s) &= -\rho D(s) ds + K(s) d\Theta_s - dR_1(s), \\ dK(s) &= \mu(K(s)) ds + \sigma(K(s)) dW_s - dR_2(s). \end{aligned}$$

Proof. Start by considering S . Since \hat{S}_3 is an $\hat{\mathcal{F}}$ -martingale, we can conclude by the optional stopping theorem and for $0 \leq s_1 \leq s_2, n \in \mathbb{N}$ that

$$\mathbb{E} \left[\hat{S}_3(T(s_2) \wedge n) \mid \hat{\mathcal{F}}(T(s_1)) \right] = \mathbb{E} \left[\hat{S}_3(T(s_2) \wedge n) \mid \hat{\mathcal{F}}(T(s_1) \wedge n) \right] = \hat{S}_3(T(s_1) \wedge n).$$

As in the proof of Lemma 3.1.13, one can show that $\{\hat{S}_3(T(s) \wedge n), n \in \mathbb{N}\}$ is uniformly integrable. Therefore,

$$\lim_{n \rightarrow \infty} \hat{S}_3(T(s) \wedge n) = \hat{S}_3(T(s)) \quad \text{in } \mathcal{L}^1$$

and $\mathbb{E} [S_3(s_2) | \mathcal{F}^0(s_1)] = S_3(s_1)$, i.e. S_3 is a martingale.

It is not obvious that S_3 is continuous, since T is left-continuous. But since S_3^h is not control dependent, we do not have to use the time rescaling. Instead, we argue directly for S_3^h as in Lemma 3.1.12 that $\{S_3^h, h > 0\}$ is tight. We call its subsequential limit \tilde{S}_3 . It is continuous analogously to Lemma 3.1.13. Let $(\tilde{S}_3, \hat{S}_3, \hat{T})$ be the subsequential limit of $(S_3^h, \hat{S}_3^h := S_3^h \circ \hat{T}^h, \hat{T}^h)$. It follows that $\hat{S}_3 = \tilde{S}_3 \circ \hat{T}$ and therefore $S_3 := \hat{S}_3 \circ T = \tilde{S}_3$ must be continuous.

As before, one can show that

$$\left\{ \left(\hat{S}_3(T(s) \wedge n) \right)^2, n \in \mathbb{N} \right\} \quad \text{and} \quad \left\{ \left\langle \hat{S}_3 \right\rangle (T(s) \wedge n), n \in \mathbb{N} \right\}$$

are uniformly integrable and thus \mathcal{L}^1 -convergence holds for these terms as $n \rightarrow \infty$. With the optional sampling theorem applied to the martingale $(\hat{S}_3)^2 - \langle \hat{S}_3 \rangle$,

$$\begin{aligned} & \mathbb{E} \left[(S_3(s_2))^2 - \left\langle \hat{S}_3 \right\rangle (T(s_2)) \mid \mathcal{F}^0(s_1) \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\left(\hat{S}_3(T(s_2) \wedge n) \right)^2 - \left\langle \hat{S}_3 \right\rangle (T(s_2) \wedge n) \mid \hat{\mathcal{F}}(T(s_1)) \right] \\ &= \lim_{n \rightarrow \infty} \left(\hat{S}_3(T(s_1) \wedge n) \right)^2 - \left\langle \hat{S}_3 \right\rangle (T(s_1) \wedge n) = (S_3(s_1))^2 - \left\langle \hat{S}_3 \right\rangle (T(s_1)). \end{aligned}$$

That is $(S_3)^2 - \langle \hat{S}_3 \rangle \circ T$ is a martingale. Hence, the uniqueness of the quadratic variation yields $\langle S_3 \rangle = \langle \hat{S}_3 \rangle \circ T$ and (3.24) follows from (3.18). The existence of a Brownian motion W satisfying (3.25) is a direct consequence of a martingale representation result as given, e.g., in Karatzas and Shreve (2000), Theorem 3.4.2.

Let us show that

$$0 \leq \int_{[0, \infty)} \mathbb{I}_{\{D(r+) < l\}} dR_1(r) = \int_{[0, \infty)} \mathbb{I}_{\{\hat{D}(T(r+)) < l\}} d\hat{R}_1(T(r))$$

equals zero. We know from Lemma 3.1.13 that

$$\int_{[0, \infty)} \mathbb{I}_{\{\hat{D}(r+) < l\}} d\hat{R}_1(r) = 0. \quad (3.28)$$

That is problems can only arise at jump points $r \in [0, \infty)$ that exhibit reflection, i.e. $T(r+) > T(r)$ and $\hat{R}_1(T(r+)) > \hat{R}_1(T(r))$. Let us assume that D is *not* minimal reflecting, i.e. $\hat{D}(T(r+)) < l$. According to Lemma 3.1.13, \hat{D} is continuous. It is also non-decreasing on $[T(r), T(r+)]$, since the control increases \hat{D} . Therefore, $\hat{D} < l$ on $[T(r), T(r+)]$ and thus

$$\int_{[T(r), T(r+)]} \mathbb{I}_{\{\hat{D}(u+) < l\}} d\hat{R}_1(u) = \int_{[T(r), T(r+)]} d\hat{R}_1(u) > 0.$$

This contradicts (3.28). Since \hat{K} is constant on $[T(r), T(r+)]$ for all $r \in [0, \infty)$, the same argument holds for the second reflection term. Alternatively, one could get this result for R_2 by the Lipschitz continuity of the Skorokhod map and by omitting the time rescaling as done with \tilde{S}_3 above. This would exploit that K is uncontrolled.

The remaining claims follow immediately from Lemma 3.1.13. For instance, (3.27) is a consequence of (3.20) and (3.23). The stochastic differential equations for D and K result from (3.27) as well as 2. and 3. \square

3.1.8 Convergence of the value function

So far, we have focused on the state space and control processes and their convergence as $h \rightarrow 0$ as well as properties of the resulting limit processes. After these preparations, we can now handle the convergence of the value function. Our approach consists of two main steps, where the first one considers the lower limit and the second one the upper limit of the discretized value function U^h . The arguments in the proof of the lower limit in Budhiraja and Ross (2007) are not appropriate for our specific cost functional. E.g. there is no consumption rate involved in our problem, we have to account for our final cost term etc. Therefore, we have to find our own proof for the following result.

Recall that for $z^0 \in \mathbb{G}_l$, (z^h) is a sequence with $Z_0^h = z^h \in \mathbb{G}_l^{h+}$ and $\lim_{h \rightarrow 0} z^h = z^0$. In the sequel, we use the following notation, if not stated otherwise: For given controls $\{I^h, h > 0\}$, $(Z^h, B^h, S^h, \Theta^h, E^h, R^h)$ as given in Subsection 3.1.3 denote the continuous time processes corresponding to I^h in the original time scale, (\hat{I}^h, \hat{E}^h) and $\hat{H}^h = (\hat{Z}^h, \hat{T}^h, \hat{\Theta}^h, \hat{R}^h, \hat{B}^h, \hat{S}^h)$ are the corresponding rescaled processes as explained in Subsection 3.1.4 and $\hat{H} = (\hat{Z}, \hat{T}, \hat{\Theta}, \hat{R}, \hat{B}, \hat{S})$ as explained at the beginning of Subsection 3.1.6 is the corresponding limit that we have thanks to the tightness result in Subsection 3.1.5 with $H = \hat{H} \circ T$ as in Subsection 3.1.7 being the process transformed back to the original time scale.

Proposition 3.1.17. (Lower limit of the discrete value function).

$$\liminf_{h \rightarrow 0} U^h(0, z^h) \geq U_l(0, z^0)$$

Proof. Let $(I^h)_{h>0}$, $I^h \in \mathcal{A}^h(z^h)$ be a family of controls. According to Lemma 3.1.10,

$$\begin{aligned} \liminf_{h \rightarrow 0} J^h(0, z^h, I^h) &= \liminf_{h \rightarrow 0} \mathbb{E} \left[\int_{[0, \hat{\tau}_0^h)} \left(\hat{D}^h(s) + \frac{\hat{K}^h(s)}{2} \Delta \hat{\Theta}^h(s) \right) d\hat{\Theta}^h(s) \right. \\ &\quad \left. + \left(\hat{D}^h(\hat{\tau}_0^h) + \frac{\hat{K}^h(\hat{\tau}_0^h)}{2} \hat{X}^h(\hat{\tau}_0^h) \right) \hat{X}^h(\hat{\tau}_0^h) \right] \\ &\geq \mathbb{E} \left[\int_{[0, \hat{\tau}_0)} \left(\hat{D}(s) + \frac{\hat{K}(s)}{2} \Delta \hat{\Theta}(s) \right) d\hat{\Theta}(s) + \left(\hat{D}(\hat{\tau}_0) + \frac{\hat{K}(\hat{\tau}_0)}{2} \hat{X}(\hat{\tau}_0) \right) \hat{X}(\hat{\tau}_0) \right]. \end{aligned}$$

Thanks to the tightness result Lemma 3.1.12 and the Skorokhod representation as explained around (3.17), the last inequality follows from Fatou's lemma. Let us show that this expectation does not change when $\hat{\tau}_0$ is replaced by

$$\hat{\underline{\tau}}_0 := \inf \left\{ s \in [0, \infty) \mid \hat{Z}(s) \in \partial^0 \text{ or } \hat{T}(s) \geq T \right\}.$$

Lemma VI.2.5 in Jacod and Shiryaev (2003) yields $\hat{\underline{\tau}}_0 \leq \hat{\tau}_0$. We next show that if $\hat{\underline{\tau}}_0 < \hat{\tau}_0$ with positive probability, this does not change the above expectation.

For $\hat{T}(\hat{\underline{\tau}}_0) = T$, we have $\hat{T}(\hat{\tau}_0) \geq T$. Since $\hat{\tau}_0$ is defined as the limit of $\hat{\tau}_0^h$, $\hat{T}(\hat{\tau}_0) = T$ and hence $\hat{\underline{\tau}}_0 = \hat{\tau}_0$. Consider $\hat{T}(\hat{\underline{\tau}}_0) < T$. Since \hat{X} is continuous and nonincreasing, it must be less than or equal to zero on $[\hat{\underline{\tau}}_0, \infty)$. It cannot become strictly negative on $[\hat{\underline{\tau}}_0, \hat{\tau}_0)$ because $\hat{X}(\hat{\tau}_0) \geq 0$. So even if there exists a scenario ω with $\hat{\underline{\tau}}_0(\omega) < \hat{\tau}_0(\omega)$, $\hat{X} \equiv 0$ on $[\hat{\underline{\tau}}_0(\omega), \hat{\tau}_0(\omega))$.

That is it only remains to show that

$$\mathbb{E} \left[\int_{[0, \hat{\underline{\tau}}_0)} \left(\hat{D}(s) + \frac{\hat{K}(s)}{2} \Delta \hat{\Theta}(s) \right) d\hat{\Theta}(s) + \left(\hat{D}(\hat{\underline{\tau}}_0) + \frac{\hat{K}(\hat{\underline{\tau}}_0)}{2} \hat{X}(\hat{\underline{\tau}}_0) \right) \hat{X}(\hat{\underline{\tau}}_0) \right] \quad (3.29)$$

$$= \mathbb{E} \left[\int_{[0, \tau_0^l)} \left(D(s) + \frac{K(s)}{2} \Delta \Theta(s) \right) d\Theta(s) + \left(D(\tau_0^l) + \frac{K(\tau_0^l)}{2} X(\tau_0^l) \right) X(\tau_0^l) \right] \quad (3.30)$$

for τ_0^l from (3.2) and (D, X, K, Θ) defined via $(\hat{D}, \hat{X}, \hat{K}, \hat{\Theta})$ as in (3.23). Then the assertion follows with Lemma 3.1.16. Due to the construction of the time scaling, we have $\tau_0^l = \hat{T}(\hat{\underline{\tau}}_0)$. Thus, (3.30) equals

$$\begin{aligned} & \mathbb{E} \left[\int_{[0, \hat{T}(\hat{\underline{\tau}}_0))} \left(\hat{D}(T(s)) + \frac{\hat{K}(T(s))}{2} \Delta \hat{\Theta}(T(s)) \right) d\hat{\Theta}(T(s)) \right. \\ & \left. + \left(\hat{D}(T(\hat{T}(\hat{\underline{\tau}}_0))) + \frac{\hat{K}(T(\hat{T}(\hat{\underline{\tau}}_0)))}{2} \hat{X}(T(\hat{T}(\hat{\underline{\tau}}_0))) \right) \hat{X}(T(\hat{T}(\hat{\underline{\tau}}_0))) \right]. \end{aligned} \quad (3.31)$$

We would be done in case of $T(\hat{T}(\hat{\underline{\tau}}_0)) = \hat{\underline{\tau}}_0$ a.s. by use of Lebesgue's change of time formula given in Protter (2004), Theorem 45 in a left-continuous version. The scenarios with $T(\hat{T}(\hat{\underline{\tau}}_0)) < \hat{\underline{\tau}}_0$ correspond to a jump of the strategy Θ to levels x or above. The costs of this jump are taken into account in the integral term in (3.29) and in the $T(\hat{T}(\hat{\underline{\tau}}_0))$ term in (3.31). Therefore, (3.29) and (3.31) are always equal to each other. \square

Notice that the time rescaling, which we discussed in Subsection 3.1.4 to 3.1.7 and which was crucial in the proof of Proposition 3.1.17, is from now on not needed anymore. The remainder of this subsection is dedicated to the upper limit analog to Proposition 3.1.17. The idea is to take a strategy Θ^{ϵ_1} that is ϵ_1 -optimal for the continuous time problem

with initial value z^0 . The ultimate goal is to approximate Θ^{ϵ_1} by a family of discrete strategies with admissible controls $\{I^h, h > 0\}$ such that

$$U_l(0, z^0) - \epsilon_1 \geq J_l(0, z^0, \Theta^{\epsilon_1}) = \lim_{h \rightarrow 0} J^h(0, z^h, I^h) \geq \limsup_{h \rightarrow 0} U^h(0, z^h). \quad (3.32)$$

For $\epsilon_1 \searrow 0$, this proves Proposition 3.1.24, but the mentioned approximation is rather involved. First of all, we show in Lemma 3.1.20 that Θ^{ϵ_1} can without loss of generality be replaced by a so called *simple strategy*, which is bounded, piecewise constant and jumps 'in increments' as described in Definition 3.1.18. Such a simple strategy can then be approximated by controlled Markov chains, as explained in the five step scheme below. Hence, a simple strategy can be interpreted as an interstage between $\Theta^{\epsilon_1} \in \tilde{\mathcal{A}}_0$ and the controlled Markov chain belonging to I^h .

In summary, the lower limit of the discrete value function is proved by constructing from admissible controls I^h a suitable limit via time rescaling and tightness. For the upper limit, it is the other way around. We aim to approximate the strategy Θ^{ϵ_1} by an admissible control sequence I^h . There are only slight adjustments of analogous proofs as in Budhiraja and Ross (2007), since the specific form of the value function plays a secondary role compared to the strategy approximations.

Definition 3.1.18. (Simple strategy).

A strategy $\Theta \in \tilde{\mathcal{A}}_0$ is called a simple strategy if it has all the following properties:

- (i) *Finite image and piecewise constant:* There exist $\eta_1, \eta_2 > 0$, $M \in \mathbb{N}$ such that

$$\Theta(s) \in \{m\eta_1 | m = 0, \dots, M\}$$

for all $s \in [0, \infty)$ and Θ is constant on the time intervals $(j\eta_2, (j+1)\eta_2]$, $j \in \mathbb{N}$.

- (ii) *Finite dependence on Brownian motion:* There exists $\tilde{\eta}_2 > 0$ such that η_2 is an integer multiple of $\tilde{\eta}_2$ and for all $j \in \mathbb{N}$, $m \in \{0, \dots, M\}$

$$\begin{aligned} & \mathbb{P}_z [\Delta\Theta(j\eta_2) = m\eta_1 | \Theta(s), s \leq j\eta_2; W^K(s), s \leq j\eta_2] \\ &= \mathbb{P}_z [\Delta\Theta(j\eta_2) = m\eta_1 | \Theta(i\eta_2), i \leq j; W^K(n\tilde{\eta}_2), n\tilde{\eta}_2 \leq j\eta_2] \\ &=: q_{j,m}(\tau(j), z, \mathcal{W}(j)) \end{aligned} \quad (3.33)$$

with $\tau(j) := \{\Theta(i\eta_2) | i \leq j\}$, $\mathcal{W}(j) := \{W^K(n\tilde{\eta}_2) | n\tilde{\eta}_2 \leq j\eta_2\}$ and

$$q_{0,m}(z) := \mathbb{P}_z [\Delta\Theta(0) = m\eta_1].$$

- (iii) *Continuity:* The above mappings $q_{j,m}$ are continuous for $j = 0, 1, 2, \dots$, $m \in \{0, \dots, M\}$.

Remark 3.1.19. In the existing literature, $q_{j,m}$ is assumed to be continuous in its second and third component. As explained in the singular control step, our control direction is state dependent. Therefore, we also need the continuity in the first component, which will be relevant in the proof of Lemma 3.1.23.

Lemma 3.1.20. (Simple strategy approximation).

For $\Theta \in \tilde{\mathcal{A}}_0$, $z^0 \in \mathbb{G}_l$ and $\epsilon_2 > 0$, there exists a simple strategy $\Theta^{\epsilon_2} \in \tilde{\mathcal{A}}_0$ such that

$$|J_l(0, z^0, \Theta) - J_l(0, z^0, \Theta^{\epsilon_2})| < \epsilon_2.$$

Proof. We apply an η_1, η_2 grid to Θ and show that the costs of the resulting strategy Θ^{η_1, η_2} converge to the costs of Θ when we refine the grid. In order to show the stopping time convergence below, we assume without loss of generality that

$$\Theta(s) = \Theta(s)\mathbb{I}_{\{s < \tau_0\}} + [x^0 + (s - \tau_0)]\mathbb{I}_{\{\tau_0 \leq s < \tau_0 + 1\}} + (x^0 + 1)\mathbb{I}_{\{s \geq \tau_0 + 1\}}. \quad (3.34)$$

Define

$$\Theta_j^{\eta_1, \eta_2} := m\eta_1 \quad \text{if } \Theta(j\eta_2) \in [m\eta_1, (m+1)\eta_1)$$

for $j \in \mathbb{N}$, $m = 0, 1, \dots, M := \lceil \frac{l+1}{\eta_1} \rceil$, $\Theta^{\eta_1, \eta_2}(0) := 0$ and for $s \in (j\eta_2, (j+1)\eta_2]$

$$\Theta^{\eta_1, \eta_2}(s) := \Theta_j^{\eta_1, \eta_2}.$$

Consider sequences $\eta_{1,n}, \eta_{2,n}$ converging to zero as $n \rightarrow \infty$ and set $\Theta^n := \Theta^{\eta_{1,n}, \eta_{2,n}}$. Then (Z^n, Θ^n) converges (in probability) in $\mathcal{D}([0, T+1] : \mathbb{R}^3 \times [0, l+1])$ to (Z, Θ) when $n \rightarrow \infty$ and also τ_0^n , defined analogously to τ_0 , converges to τ_0 due to the construction (3.34). Using dominated convergence, this shows that the expected costs of Θ^n converge to the expected costs of Θ as desired.

For given η_1, η_2 , let us approximate Θ^{η_1, η_2} by a simple strategy $\Theta^{\check{\eta}_2}$, which satisfies the finite dependence and continuity property. To do so, let $q_{j,m}^{\eta_1, \eta_2}$ be the conditional expectations as in (3.33) that belong to Θ^{η_1, η_2} and define the dynamics of the process $\Theta^{\check{\eta}_2}$ via $\Theta^{\check{\eta}_2}(0) := 0$,

$$\begin{aligned} \mathbb{P}_z [\Delta \Theta^{\check{\eta}_2}(j\eta_2) = m\eta_1 | \Theta^{\eta_1, \eta_2}(i\eta_2), i \leq j; W^K(n\check{\eta}_2), n\check{\eta}_2 \leq j\eta_2] := \\ q_{j,m}^{\eta_1, \eta_2}(\mathcal{T}^{\eta_1, \eta_2}(j), z, \mathcal{W}(j)) \end{aligned}$$

and $\Theta^{\check{\eta}_2}$ constant on $(j\eta_2, (j+1)\eta_2]$. Due to the martingale convergence theorem, $\Theta^{\check{\eta}_2}$ converges to Θ^{η_1, η_2} as $\check{\eta}_2 \searrow 0$.

For the continuity property, the idea is to approximate $q_{j,m}$ by mollified functions. E.g. for $F : \mathbb{R} \rightarrow \mathbb{R}$ integrable, the mollified function

$$F^\vartheta(x) := \frac{1}{\sqrt{2\pi\vartheta}} \int_{-\infty}^{\infty} F(x+y) e^{-\frac{y^2}{2\vartheta}} dy = \frac{1}{\sqrt{2\pi\vartheta}} \int_{-\infty}^{\infty} F(y) e^{-\frac{(y-x)^2}{2\vartheta}} dy$$

is continuous and converges to F as $\vartheta \searrow 0$. We exemplarily go through this mollifier argument for the w -component. For $\vartheta > 0$, define

$$\begin{aligned} q_{j,m}^\vartheta(\mathcal{T}(j); z; w(n\check{\eta}_2), n\check{\eta}_2 \leq j\eta_2) := \\ N(\vartheta) \int \dots \int q_{j,m} \left(\mathcal{T}(j), z, (w(n\check{\eta}_2) + y_n)_{n=0, \dots, \frac{j\eta_2}{\check{\eta}_2}} \right) \prod_{n=0}^{\frac{j\eta_2}{\check{\eta}_2}} \left(e^{-\frac{y_n^2}{2\vartheta}} dy_n \right). \end{aligned}$$

for the normalizer $N(\vartheta)$ such that $\sum_{m=0}^M q_{j,m}^\vartheta = 1$. Define the piecewise constant control Θ^ϑ by setting its corresponding conditional probability to q^ϑ . Due to the piecewise constant nature of both Θ and Θ^ϑ , the corresponding costs $J_l(0, z^0, \Theta)$, $J_l(0, z^0, \Theta^\vartheta)$ are the expectation of the sum of the cost of the trades at $j\eta_2$. Due to the construction, $\Delta\Theta(j\eta_2)$ and $\Delta\Theta^\vartheta(j\eta_2)$ have the same probability law in the limit and hence also the costs converge as $\vartheta \searrow 0$. \square

Thanks to Lemma 3.1.20, we can without loss of generality assume the strategy Θ^{ϵ_1} in (3.32) to be a simple strategy. That is we take $\eta_1, \eta_2, M, \check{\eta}_2, q_{j,m}(\cdot)$ corresponding to Θ^{ϵ_1} as given and use these quantities to define discrete strategies

$$(I_n^h, Z_n^h)_{n \in \mathbb{N}}$$

by the following five step scheme. Notice that this scheme is designed for the convergence proof only and does not have anything to do with the implementation.

Construction of discrete strategies for given simple strategy and fixed $h > 0$

1. *Initialization:* Set $\tilde{n}_0 := 0$, $Z_0^h := z^h$, draw $m \in \{0, \dots, M\}$ with probability $q_{0,m}(z^h)$ and put $\tau^h(1) := \{m\eta_1\}$.
2. *Control with reflection:* If $m = 0$, set the number of control steps to $n_1 := 0$. Otherwise define $(I_n^h, Z_{n+1}^h)_{n=0, \dots, n_1-1}$ as follows:
Set $I_0^h := 1$. Draw Z_1^h from $p^h(Z_0^h, I_0^h, \cdot)$. If $Z_1^h \in \partial_R^h$ set $I_1^h := 2$, otherwise $I_1^h := 1$. Draw Z_2^h from $p^h(Z_1^h, I_1^h, \cdot)$ etc. until $\lfloor m\eta_1(\frac{h}{\kappa^h \sqrt{V}})^{-1} \rfloor$ control steps have been done. Let n_1 denote the total number of control and reflection steps needed so far.
3. *Diffusion with reflection:*
Let $\tilde{n}_1 := \inf\{n | t_n^h \geq \eta_2\}$ and define $(I_n^h, Z_{n+1}^h)_{n=n_1, \dots, \tilde{n}_1-1}$ as follows:
If $Z_{n_1}^h \in \partial_R^h$ set $I_{n_1}^h := 2$, otherwise $I_{n_1}^h := 0$. Draw $Z_{n_1+1}^h$ from $p^h(Z_{n_1}^h, I_{n_1}^h, \cdot)$ etc.
4. *Brownian motion:*
For $n = \tilde{n}_0, \dots, \tilde{n}_1 - 1$ and independent, standard normally distributed ν_n , define

$$W_n^h := \sum_{i=0}^{n-1} \left\{ \frac{S_{i+1,3}^h - S_{i,3}^h}{\sigma(K_i^h)} \mathbb{I}_{\{\sigma(K_i^h) \neq 0\}} + \nu_i \sqrt{\Delta_i^h} \mathbb{I}_{\{\sigma(K_i^h) = 0\}} \right\}$$

with S^h obtained from Z^h as in (3.10). This choice of W^h is motivated by (3.25). For $s \in (0, \eta_2]$, $W^h(s) := W_{n^h(s)}^h - W_{\tilde{n}_0}^h$ and $w^h(1) := \{W^h(n\check{\eta}_2) | n\check{\eta}_2 \leq \eta_2\}$.

5. *Start next loop:* For $\tilde{n}_k := \inf\{n | t_n^h \geq k\eta_2\}$, let $(I_n^h, Z_{n+1}^h)_{n=0, \dots, \tilde{n}_k-1}, \tau^h(k), w^h(k)$ be given. Draw m from $q_{k,m}(\tau^h(k), Z_{\tilde{n}_k}^h, w^h(k))$ and $\tau^h(k+1) := \tau^h(k) \cup \{m\eta_1\}$. Continue with Step 2 with the coefficients being adjusted to $k+1$.

In Lemma 3.1.21, we show that the law of the continuous time interpolation (Z^h, W^h) converges weakly to the law of (Z^{ϵ_1}, W^K) on a fixed interval $[j\eta_2, (j+1)\eta_2]$ of the simple

strategy as $h \rightarrow 0$. Having no control instances on the inner part of this interval is the trick to get tightness without having to use the time rescaling method. That is for a simple strategy with $Z(0) = (\delta, x, \kappa) \in \mathbb{G}_l$ and $s \in (0, \eta_2]$, there exists $m \in \{0, \dots, M\}$ such that

$$\begin{aligned} \Theta(s) &\equiv m\eta_1, & D(s) &= [(\delta + \kappa m\eta_1) \wedge l] e^{-\rho s}, & X(s) &\equiv x - m\eta_1, \\ K(s) &= \kappa + \int_{[0,s)} \mu(K(r)) dr + \int_{[0,s)} \sigma(K(r)) dW^K(r) - \int_{[0,s)} dR_2(r). \end{aligned}$$

Lemma 3.1.21. (Weak convergence on no control period).

Let Θ be a simple strategy with corresponding (Z, W^K) . Set $Z_0^h := z^h$ and define $(I_n^h, Z_{n+1}^h)_{n=0, \dots, \tilde{n}_1-1}$ as in the five step scheme above. Replace Δ_0^h by $\Delta_0^h + \epsilon^h$ with $\epsilon^h \searrow 0$ and $Z^h(s) := Z_{n^h(s)}^h$. Denote the laws of (Z^h, W^h) and (Z, W^K) on $\mathcal{D}([0, \eta_2] : \mathbb{R}^4)$ by Π_{m, δ^h}^h and Π_m . Then $\Pi_{m, \delta^h}^h \xrightarrow{w} \Pi_m$ as $h \rightarrow 0$, where h indexes a suitable subsequence.

Remark 3.1.22. We need to allow for ϵ^h in Lemma 3.1.21, since it can happen in Step 3 of our scheme that $t_{\tilde{n}_k}^h > k\eta_2$. Hence, the following control step sits at $t_{\tilde{n}_k}^h$ instead of $k\eta_2$. The ϵ^h corresponds to the difference $(t_{\tilde{n}_k}^h - k\eta_2)$.

Proof. The sequence of distributions corresponding to the process W^h from Step 4 is tight as one can show as in Lemma 3.1.12. The limit of W^h must be a Brownian motion as can be shown, e.g., as in 3. of Lemma 3.1.13 together with Lévy's condition.

Since there is no control in the interior of the considered time interval and Z^h is constructed in Step 2 and 3 according to the diffusion and control consistency conditions, it is clear that the weak convergence holds for Z^h . \square

In Lemma 3.1.23, we exploit Lemma 3.1.21 to conclude that the costs of a simple strategy are equal to the costs of the corresponding I^h from the five step scheme in the limit. We can then conclude in Proposition 3.1.24 that (3.32) holds.

Lemma 3.1.23. (Cost convergence for the five step scheme).

Let Θ be a simple strategy and take $\{I^h, h > 0\}$ from the five step scheme. Then

$$J_l(0, z^0, \Theta) = \lim_{h \rightarrow 0} J^h(0, z^h, I^h).$$

Proof. Let us have a closer look at Θ^h corresponding to I^h from the five step scheme. In Step 2 and thanks to the continuity property of a simple strategy, we see that the conditional jump distribution of Θ^h is close to the jump distribution of the simple strategy Θ . According to Step 3 and as explained in Remark 3.1.22, the jump times of Θ^h are close to the jump times $\{0, \eta_2, 2\eta_2, \dots\}$ of Θ . It follows that $\Theta^h \xrightarrow{w} \Theta$ on $\mathcal{D}([0, \infty) : \mathbb{R})$. As we have seen in Lemma 3.1.21, we also have convergence of (Z^h, W^h) to (Z, W^K) between the jump times. Cost convergence then follows as in the proof of Lemma 3.1.20. \square

Proposition 3.1.24. (Upper limit of the discrete value function).

$$\limsup_{h \rightarrow 0} U^h(0, z^h) \leq U_l(0, z^0)$$

Proof. The claim follows immediately from (3.32) together with Lemma 3.1.20 and the cost convergence from Lemma 3.1.23. \square

As a direct consequence of Proposition 3.1.17 and 3.1.24, we obtain the main result of this section.

Theorem 3.1.25. (Convergence of the value function as mesh size decreases).

$$\lim_{h \rightarrow 0} U^h(0, z^h) = U_l(0, z^0)$$

It was our main purpose to show this convergence theorem. As an additional consequence of our considerations, we get existence of an optimal strategy for the truncated value function in the following corollary. Such an existence result is not contained in Kushner and Dupuis (2001), Kushner and Martins (1991) or Budhiraja and Ross (2007). It may be of interest in our problem, since we needed additional convexity assumptions in order to prove existence for the original, non-truncated problem in Proposition 2.4.3.

Corollary 3.1.26. (Existence of an optimal strategy for the truncated value function).
There exists $\Theta^l \in \tilde{\mathcal{A}}_0$ such that

$$U_l(0, z^0) = J_l(0, z^0, \Theta^l).$$

Proof. Our aim is to show that there exists a sequence of controls $(I^{h_n})_{n \in \mathbb{N}}$ with $h_n \searrow 0$ and

$$U_l(0, z^0) = \lim_{n \rightarrow \infty} J^{h_n}(0, z^{h_n}, I^{h_n}). \quad (3.35)$$

Following the same tightness argument as in the proof of Proposition 3.1.17, we then get the existence of $\Theta^l \in \tilde{\mathcal{A}}_0$ such that the right-hand side equals $J_l(0, z^0, \Theta^l)$ and we are done. More precisely, we define the controlled Markov chain Z^{h_n} using I^{h_n} and do the time rescaling, which yields the corresponding \hat{H}^{h_n} . The tightness Lemma 3.1.12 then guarantees the existence of \hat{H} such that $\hat{H}^{h_n} \xrightarrow{w} \hat{H}$ and we can undo the time rescaling to get H containing Θ^l and having the nice properties given in Lemma 3.1.16.

It only remains to show (3.35). Let us take a minimizing sequence of strategies $\Theta^\epsilon \in \tilde{\mathcal{A}}_0$ such that

$$U_l(0, z^0) = \lim_{\epsilon \rightarrow 0} J_l(0, z^0, \Theta^\epsilon).$$

Due to Lemma 3.1.20, we can assume each Θ^ϵ to be a simple strategy. For a given simple strategy Θ^ϵ , we can use the five step scheme to define Markov chain controls $(I^{h,\epsilon})_{h>0}$ such that Lemma 3.1.23 guarantees

$$J_l(0, z^0, \Theta^\epsilon) = \lim_{h \rightarrow 0} J^h(0, z^h, I^{h,\epsilon}). \quad (3.36)$$

We do a diagonalization argument. Consider a sequence $(\epsilon_k)_{k \in \mathbb{N}}$ with $\epsilon_k \searrow 0$ as $k \rightarrow \infty$. For $k = 1$ and according to (3.36), there exists $h_n^{(1)} \searrow 0$ such that

$$J_l(0, z^0, \Theta^{\epsilon_1}) = \lim_{n \rightarrow \infty} J^{h_n^{(1)}}(0, z^{h_n^{(1)}}, I^{h_n^{(1)}, \epsilon_1}).$$

The same equation is true for ϵ_2 and a subsequence $(h_n^{(2)})$ of $(h_n^{(1)})$. Ergo, taking the diagonal sequence $(h_n^{(n)})$ yields

$$U_l(0, z^0) = \lim_{\epsilon \rightarrow 0} J_l(0, z^0, \Theta^\epsilon) = \lim_{n \rightarrow \infty} J^{h_n^{(n)}}(0, z^{h_n^{(n)}}, I^{h_n^{(n)}, \epsilon_n})$$

as desired in (3.35). □

3.1.9 Dynamic programming equation

Thanks to the state space truncation in Proposition 3.1.1 and the convergence in Theorem 3.1.25, we can fix a large $l > 0$ for the grid range and a small $h > 0$ for the mesh size of the grid and approximate the value function by calculating U^h as given in (3.7). Assume for simplicity that l is a multiple of h and T is a multiple of the time step dt^h . In the following, we go backwards in time and use dynamic programming in order to derive an algorithm for the calculation of U^h on the grid.

Let us give a preliminary remark concerning reflection. Due to normal reflection of the controlled Markov chain, we get for $z \in \partial_R^h \setminus \partial^h = \{(l+h, x, \kappa), (\delta, x, l+h)\}$, that $I_0^h = 2$ and

$$U^h(t, (l+h, x, \kappa)) = U^h(t, (l, x, \kappa)), \quad U^h(t, (\delta, x, l+h)) = U^h(t, (\delta, x, l)).$$

As boundary conditions, we get from (3.7) that

$$\begin{aligned} U^h(t, (\delta^h, 0, \kappa^h)) &= 0 \quad \text{and} \\ U^h(T, z^h) &= \left(\delta^h + \frac{\kappa^h}{2} x^h \right) x^h. \end{aligned}$$

For $t \in \{0, dt^h, \dots, T - dt^h\}$, $z^h \in \mathbb{G}_l^h$, we can take a diffusion or control step. Hence,

$$\begin{aligned}
U^h(t, z^h) &= \min \left\{ \sum_{z' \in \mathbb{G}_l^{h+}} p^h(z^h, 0, z') U^h(t + \Delta^h(z^h, 0), z'), \right. \\
&\quad \left. \left(\delta^h + \frac{\kappa^h h}{2 \kappa^h \vee 1} \right) \frac{h}{\kappa^h \vee 1} + \sum_{z' \in \mathbb{G}_l^{h+}} p^h(z^h, 1, z') U^h(t + \Delta^h(z^h, 1), z') \right\} \\
&= \min \{ q_{(0)}^h(z^h, z^h) U^h(t + dt^h, z^h) \\
&\quad + q_{(0)}^h(z^h, (\delta^h - h, x^h, \kappa^h)) U^h(t + dt^h, (\delta^h - h, x^h, \kappa^h)) \\
&\quad + q_{(0)}^h(z^h, (\delta^h, x^h, \kappa^h - h)) U^h(t + dt^h, (\delta^h, x^h, \kappa^h - h)) \\
&\quad + q_{(0)}^h(z^h, (\delta^h, x^h, \kappa^h + h)) U^h(t + dt^h, (\delta^h, x^h, (\kappa^h + h) \wedge l)), \\
&\quad \left(\delta^h + \frac{\kappa^h h}{2 \kappa^h \vee 1} \right) \frac{h}{\kappa^h \vee 1} \\
&\quad + q_{(1)}^h(z^h, (\delta^h, x^h - h, \kappa^h)) U^h(t, (\delta^h, x^h - h, \kappa^h)) \\
&\quad + q_{(1)}^h(z^h, (\delta^h + h, x^h, \kappa^h)) U^h(t, ((\delta^h + h) \wedge l, x^h, \kappa^h)) \\
&\quad + q_{(1)}^h(z^h, (\delta^h + h, x^h - h, \kappa^h)) U^h(t, ((\delta^h + h) \wedge l, x^h - h, \kappa^h)) \}.
\end{aligned} \tag{3.37}$$

This equation induces an explicit scheme for the computation of $U^h(t, z^h)$ by a nested for-loop over

$$\begin{aligned}
t &= T - dt^h, \dots, 0 \\
\kappa^h &= 0, h, \dots, l \\
\delta^h &= l, l - h, \dots, 0 \\
x^h &= h, 2h, \dots, l.
\end{aligned} \tag{3.38}$$

In each step, we calculate the term corresponding to diffusion and control in (3.37), compare these two terms and set $U^h(t, z^h)$ to the minimum. The order (3.38) in the for-loop is chosen such that we always know U^h at all neighboring points that are necessary for this calculation. Since we run through all state dimensions (δ, x, κ) separately, this algorithm has complexity

$$O\left(\frac{T}{dt^h} \left[\frac{l}{h}\right]^3\right).$$

Recall the dimension reduction from Lemma 1.3.1. It would be nice to reduce the complexity of the algorithm by combining the x and δ dimension. This is the topic of the next subsection.

3.1.10 Complexity-reduced dynamic programming equation

The aim of this subsection is to modify the numerical scheme (3.37) such that it has two instead of three state space dimensions. To do so, we need to state the scaling Lemma 1.3.1 as a version for the value function U^h on the grid.

Lemma 3.1.27. (Dimension reduction for the discrete value function).

Assume that one is an integer multiple of h , $t \in [0, T]$ and $z^h \in \mathbb{G}_l^h$. Let $v \in \mathbb{N}$ be the least common multiple of δ^h and one, as well as $v \vee \lceil \frac{x^h/h}{\delta^h} \rceil hv \leq l$. Then

$$U^h \left(t, \left(1, \left\lfloor \frac{x^h/h}{\delta^h} \right\rfloor h, \kappa^h \right) \right) \leq (\delta^h)^{-2} U^h (t, z^h) \leq U^h \left(t, \left(1, \left\lceil \frac{x^h/h}{\delta^h} \right\rceil h, \kappa^h \right) \right), \quad (3.39)$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ denote, respectively, the nearest integers smaller and larger than x .

Proof. From the definition (3.7) of J^h , we see that (1.21) holds analogously, but we need to take care to stay on the grid \mathbb{G}_l^h . We have $\frac{v}{\delta^h} \in \mathbb{N}$ and dimension reduction yields

$$\left(\frac{v}{\delta^h} \right)^2 U^h (t, (\delta^h, x^h, \kappa^h)) = U^h \left(t, \left(v, \frac{v}{\delta^h} x^h, \kappa^h \right) \right)$$

as well as

$$v^2 U^h \left(t, \left(1, \left\lfloor \frac{x^h/h}{\delta^h} \right\rfloor h, \kappa^h \right) \right) = U^h \left(t, \left(v, \left\lfloor \frac{x^h/h}{\delta^h} \right\rfloor hv, \kappa^h \right) \right)$$

and

$$v^2 U^h \left(t, \left(1, \left\lceil \frac{x^h/h}{\delta^h} \right\rceil h, \kappa^h \right) \right) = U^h \left(t, \left(v, \left\lceil \frac{x^h/h}{\delta^h} \right\rceil hv, \kappa^h \right) \right).$$

Plugging this into the assertion (3.39) yields

$$U^h \left(t, \left(v, \left\lfloor \frac{x^h/h}{\delta^h} \right\rfloor hv, \kappa^h \right) \right) \leq U^h \left(t, \left(v, \frac{v}{\delta^h} x^h, \kappa^h \right) \right) \leq U^h \left(t, \left(v, \left\lceil \frac{x^h/h}{\delta^h} \right\rceil hv, \kappa^h \right) \right),$$

since

$$\left\lfloor \frac{x^h/h}{\delta^h} \right\rfloor hv \leq \frac{v}{\delta^h} x^h \leq \left\lceil \frac{x^h/h}{\delta^h} \right\rceil hv.$$

□

We see from (3.39) that it suffices to calculate $U^h(t, (1, x^h, \kappa^h))$ and this gives us an approximation for U^h on the whole grid. That is we can set $\delta^h = 1$ in the dynamic programming equation (3.37). The right-hand side of (3.37) then contains, e.g., the term

$$U^h (t + dt^h, (1 - h, x^h, \kappa^h)).$$

If $(1, x^h/(1 - h), \kappa^h)$ is not a grid point, i.e. the left-hand and the right-hand term of (3.39) are unequal, we need to approximate this term by a linear interpolation using the neighboring grid points

$$\left(1, \left\lfloor \frac{x^h/h}{1-h} \right\rfloor h, \kappa^h \right) \text{ and } \left(1, \left\lceil \frac{x^h/h}{1-h} \right\rceil h, \kappa^h \right).$$

That is

$$\begin{aligned}
U^h(t + dt^h, (1 - h, x^h, \kappa^h)) &= (1 - h)^2 \tag{3.40} \\
&\left\{ O(h^2) + U^h\left(t + dt^h, \left(1, \left\lfloor \frac{x^h/h}{1-h} \right\rfloor h, \kappa^h\right)\right) + \frac{1}{h} \left(\frac{x^h}{1-h} - \left\lfloor \frac{x^h/h}{1-h} \right\rfloor h\right) \right. \\
&\left. \left[U^h\left(t + dt^h, \left(1, \left\lceil \frac{x^h/h}{1-h} \right\rceil h, \kappa^h\right)\right) - U^h\left(t + dt^h, \left(1, \left\lfloor \frac{x^h/h}{1-h} \right\rfloor h, \kappa^h\right)\right) \right] \right\}.
\end{aligned}$$

Due to $x^h/(1-h) > x^h$, our results are only reliable for $x^h < l(1-h)^n$, where n stands for the number of iterations. That is for fixed h , we need to choose l quite large in the dimension-reduced case in order to get reliable results for small values of x^h . This is a disadvantage of the dimension-reduced Kushner implementation compared to the direct implementation of the dimension-reduced HJB variational inequality, that we heuristically present in the next subsection.

The $O(h^2)$ error due to the linear interpolation is negligible for small h . Hence,

$$\begin{aligned}
U^h(t, (1, x^h, \kappa^h)) &= \min \{ q_{(0)}^h(z^h, z^h) U^h(t + dt^h, z^h) \tag{3.41} \\
&+ q_{(0)}^h(z^h, (1 - h, x^h, \kappa^h)) \mathbf{U}^h(t + dt^h, (1 - h, x^h, \kappa^h)) \\
&+ q_{(0)}^h(z^h, (1, x^h, \kappa^h - h)) U^h(t + dt^h, (1, x^h, \kappa^h - h)) \\
&+ q_{(0)}^h(z^h, (1, x^h, \kappa^h + h)) U^h(t + dt^h, (1, x^h, (\kappa^h + h) \wedge l)), \\
&\left(1 + \frac{\kappa^h}{2} \frac{h}{\kappa^h \vee 1}\right) \frac{h}{\kappa^h \vee 1} \\
&+ q_{(1)}^h(z^h, (1, x^h - h, \kappa^h)) U^h(t, (1, x^h - h, \kappa^h)) \\
&+ q_{(1)}^h(z^h, (1 + h, x^h, \kappa^h)) \mathbf{U}^h(t, (1 + h, x^h, \kappa^h)) \\
&+ q_{(1)}^h(z^h, (1 + h, x^h - h, \kappa^h)) \mathbf{U}^h(t, (1 + h, x^h - h, \kappa^h)) \}
\end{aligned}$$

with $z^h = (1, x^h, \kappa^h)$ is the dimension-reduced dynamic programming equation. The boldly-printed terms need to be replaced by a linear interpolation as explained in (3.40). The complexity of this algorithm reduces to

$$O\left(\frac{T}{dt^h} \left[\frac{l}{h}\right]^2\right).$$

3.2 Finite difference method

As an alternative to the Kushner method from Section 3.1, we now want to discuss the solution of variational inequality (1.25) using an *explicit finite difference method*, which we derive and explain in the first part of this section. This discussion will be heuristic, i.e. we neither give a verification argument nor do we prove that the value function

from the numerical scheme converges to a solution of (1.25). Anyhow, it turns out that stability issues arise. Therefore, we need to be careful with the choice of the grid and model parameters. The second part of this section is devoted to the specification of necessary conditions for the stability of this numerical scheme. These conditions are similar to the condition in Section 3.1 to have positive transition probabilities for the controlled Markov chain. We will also see heuristically that the Kushner implementation from Section 3.1 and the implementation of the HJB variational inequality correspond to each other.

3.2.1 Explicit finite difference scheme

For an introduction to finite difference methods we refer to Seydel (2004), Chapter 4. It is the most elementary approach to the numerical solution of a differential equations. The idea is to approximate the value function on an equidistant grid. We know the value function at T . Replacing the derivatives in the variational inequality (1.25) by difference quotients, we can calculate the value function at $T - \Delta t$ etc. Let us explain this approach in more detail.

Fix $0 < y_{max}$, $0 < \kappa_{min} < \kappa_{max}$ and consider the value function on $[0, T] \times [0, y_{max}] \times [\kappa_{min}, \kappa_{max}]$. Choose step sizes $\Delta t, \Delta y, \Delta \kappa > 0$ and look for approximations $v_{i,j}^n$ of the value function $V(n\Delta t, i\Delta y, \kappa_{min} + j\Delta \kappa)$ on an equidistant grid with

$$\begin{aligned} M_t &:= \frac{T}{\Delta t}, & n &= 0, \dots, M_t, \\ M_y &:= \frac{y_{max}}{\Delta y}, & i &= 0, \dots, M_y, \\ M_\kappa &:= \frac{\kappa_{max} - \kappa_{min}}{\Delta \kappa}, & j &= 0, \dots, M_\kappa. \end{aligned}$$

As boundary condition, we have $v_{0,j}^n = 0$ for all n, j . Define

$$\rho^n := \rho_{n\Delta t}, \quad \mu_j^n := \mu(n\Delta t, \kappa_{min} + j\Delta \kappa), \quad \sigma_j^n := \sigma(n\Delta t, \kappa_{min} + j\Delta \kappa).$$

We work backward in time using the terminal condition

$$v_{i,j}^{M_t} := i\Delta y + \frac{1}{2} (\kappa_{min} + j\Delta \kappa) (i\Delta y)^2,$$

which is valid for all $i = 0, \dots, M_y, j = 0, \dots, M_\kappa$.

Consider the partial differential equation $\mathcal{W}(V) = 0$. Replacing the time derivative in this PDE by a backward difference quotient and the spatial derivatives by central differences yields

$$\begin{aligned} & \frac{v_{i,j}^n - v_{i,j}^{n-1}}{\Delta t} - 2\rho^n v_{i,j}^n + \rho^n i \Delta y \frac{v_{i+1,j}^n - v_{i-1,j}^n}{2\Delta y} \\ & + \mu_j^n \frac{v_{i,j+1}^n - v_{i,j-1}^n}{2\Delta \kappa} + \frac{1}{2} (\sigma_j^n)^2 \frac{v_{i,j+1}^n + v_{i,j-1}^n - 2v_{i,j}^n}{(\Delta \kappa)^2} = 0. \end{aligned}$$

For $i = 1, \dots, M_y - 1$ and $j = 1, \dots, M_\kappa - 1$, this motivates us to take the following approximation in the wait region:

$$\begin{aligned} \tilde{v}_{i,j}^{n-1} &:= v_{i,j}^n (1 - 2\rho^n \Delta t) \\ &+ \Delta t \left(\rho^n i \Delta y \frac{v_{i+1,j}^n - v_{i-1,j}^n}{2\Delta y} + \mu_j^n \frac{v_{i,j+1}^n - v_{i,j-1}^n}{2\Delta \kappa} + \frac{1}{2} (\sigma_j^n)^2 \frac{v_{i,j+1}^n + v_{i,j-1}^n - 2v_{i,j}^n}{(\Delta \kappa)^2} \right). \end{aligned}$$

For $j = 0$ ($i = M_y$ or $j = M_\kappa$), we need to take a forward (backward) instead of a central difference. Notice that we have constructed a two level *explicit finite difference scheme*, meaning that we can calculate \tilde{v}^{n-1} directly from v^n without having to solve a system of equations.

Consider the partial differential equation $\mathcal{B}(V) = 0$. Accordingly,

$$1 + 2(\kappa_{min} + j\Delta\kappa) v_{i-1,j}^{n-1} - \left(1 + (\kappa_{min} + j\Delta\kappa) \left(i - \frac{1}{2} \right) \Delta y \right) \frac{v_{i,j}^{n-1} - v_{i-1,j}^{n-1}}{\Delta y} = 0.$$

For $i = 1, \dots, M_y$ and $j = 0, \dots, M_\kappa$, this motivates

$$\hat{v}_{i,j}^{n-1} := v_{i-1,j}^{n-1} + \Delta y \frac{1 + 2(\kappa_{min} + j\Delta\kappa) v_{i-1,j}^{n-1}}{1 + (\kappa_{min} + j\Delta\kappa) \left(i - \frac{1}{2} \right) \Delta y}.$$

In order to find out if it is optimal to wait or buy, we set

$$v_{i,j}^{n-1} := \min \{ \tilde{v}_{i,j}^{n-1}, \hat{v}_{i,j}^{n-1} \}.$$

Summary of the algorithm

Set initial condition $v_{i,j}^{M_t} \forall i, j$

For $n = M_t, \dots, 1$

Set boundary condition $v_{0,j}^{n-1} = 0 \forall j$ and $BR^{n-1} = \emptyset$

Calculate $\tilde{v}_{i,j}^{n-1} \forall i, j$ from v^n

For $j = 0, \dots, M_\kappa$

For $i = 1, \dots, M_y$

Calculate $\hat{v}_{i,j}^{n-1}$ from $v_{i-1,j}^{n-1}$

If $\tilde{v}_{i,j}^{n-1} > \hat{v}_{i,j}^{n-1}$

$v_{i,j}^{n-1} = \hat{v}_{i,j}^{n-1}$ and $BR^{n-1} = BR^{n-1} \cup \{(i\Delta y, \kappa_{min} + j\Delta\kappa)\}$

Else $v_{i,j}^{n-1} = \tilde{v}_{i,j}^{n-1}$

3.2.2 Finite difference linked to Markov chain method

Let us demonstrate heuristically that the numerical scheme, that we derived using the Kushner method, can be understood as an explicit finite difference scheme of the variational inequality (1.24). We use U_+^{h,δ^+,x^-} etc. as a short hand notation for

$$U^h(t + dt^h, (\delta^h + h, x - h, \kappa^h)) \text{ etc.}$$

Ignoring reflection and plugging the diffusion and control transition probabilities $q_{(0)}^h$ and $q_{(1)}^h$ explicitly into the dynamic programming equation (3.37) yields

$$\min \left\{ \begin{aligned} & \frac{U_+^h - U^h}{dt^h} - \rho \delta^h \frac{U_+^h - U_+^{h,\delta^-}}{h} + \mu (\kappa^h)^+ \frac{U_+^{h,\kappa^+} - U_+^h}{h} - \mu (\kappa^h)^- \frac{U_+^h - U_+^{h,\kappa^-}}{h} \\ & + \frac{1}{2} \sigma^2 (\kappa^h) \frac{U_+^{h,\kappa^+} + U_+^{h,\kappa^-} - 2U_+^h}{h^2}, \\ & \left\{ \begin{aligned} & \kappa^h \frac{U^{h,\delta^+,x^-} - U^{h,x^-}}{h} - \frac{U^h - U^{h,x^-}}{h} + \delta^h + \frac{\kappa^h}{2} h \quad \text{for } \kappa^h \leq 1 \\ & \kappa^h \frac{U^{h,\delta^+} - U^h}{h} - \frac{U^{h,\delta^+} - U^{h,\delta^+,x^-}}{h} + \delta^h + \frac{1}{2} h \quad \text{otherwise} \end{aligned} \right\} \right\} = 0. \end{aligned}$$

That is for small h , (3.37) is an equation that we can get by replacing the derivatives in the HJB variational inequality

$$\min \left\{ \partial_t U - \rho \delta \partial_\delta U + \mu \partial_\kappa U + \frac{1}{2} \sigma^2 \partial_{\kappa\kappa} U, \kappa \partial_\delta U - \partial_x U + \delta \right\} = 0$$

by difference quotients. In fact, we have chosen the diffusion transition probabilities $q_{(0)}^h$ specifically such that this correspondence to the HJB equation holds.

3.2.3 Stability of an initial-value problem

In general, explicit finite difference schemes as introduced in Subsection 3.2.1 and 3.2.2 are easy to derive and implement, but one needs to be careful when choosing the parameters such as the step sizes to get approximations that converge to the solution of the considered PDE. In this subsection, we discuss this stability issue at least for a scheme of an initial-value problem which is similar to our schemes above. The material of this subsection is taken from Thomas (1995).

Take the following *initial-value problem* as an illustration: For constant coefficients $\delta, \delta_y, \delta_\kappa, \delta_{\kappa\kappa}$, let $G : [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ with initial condition $G(0, y, \kappa) = y + \frac{\kappa}{2} y^2$ and

$$\partial_t G + \delta G + \delta_y (\partial_y G) + \delta_\kappa (\partial_\kappa G) + \delta_{\kappa\kappa} (\partial_{\kappa\kappa} G) = 0. \quad (3.42)$$

This initial-value problem is similar to our wait region PDE. In practice, there are four aspects that make stability for the algorithm explained in Subsection 3.2.1 more complicated than for this simple initial-value problem. First, the implementation of the algorithm is done for the truncated state space $[0, y_{max}] \times [\kappa_{min}, \kappa_{max}]$. That is we cannot use the norm given in Definition 3.2.1 below, but would have to deal with sequences of norms. Second, we have to take the boundary condition $V(t, 0, \kappa) = 0$ into account. However, the stability conditions, that we will get for the simple initial-value problem, are still necessary, when we truncate the state space and set boundary conditions. Third, the coefficients of $\mathcal{W}(V) = 0$ are not constant. This would spoil calculation (3.45). Therefore, we can only check local stability at every point (t, y, κ)

with the method described in this subsection. Fortunately, according to Richtmyer and Morton (1994), Chapter 5, practical experience shows that instabilities usually start as local phenomena. As a fourth difficulty, we are not aware of any discussion in the literature concerning the stability of a difference scheme, when two PDEs are combined as in our variational inequality.

For $n \in \mathbb{N}$, $i, j \in \mathbb{Z}$, derive an explicit finite difference scheme for (3.42)

$$g_{i,j}^{n+1} = g_{i,j}^n (1 - \delta \Delta t) - \Delta t \left(\delta_y \frac{g_{i+1,j}^n - g_{i-1,j}^n}{2\Delta y} + \delta_\kappa \frac{g_{i,j+1}^n - g_{i,j-1}^n}{2\Delta \kappa} + \delta_{\kappa\kappa} \frac{g_{i,j+1}^n + g_{i,j-1}^n - 2g_{i,j}^n}{(\Delta \kappa)^2} \right). \quad (3.43)$$

Define the infinite dimensional sequence space in two dimensions with the two norm

$$l_2 := \left\{ \mathbf{g} = (g_{i,j})_{i,j \in \mathbb{Z}} : \sum_{i,j=-\infty}^{\infty} |g_{i,j}|^2 < \infty \right\}, \quad \|\mathbf{g}\|_{l_2} := \sqrt{\sum_{i,j=-\infty}^{\infty} |g_{i,j}|^2 \Delta y \Delta \kappa}.$$

Let us discuss the issue of convergence of a numerical scheme in more detail. Since it is difficult to establish convergence directly, one uses Lax's Equivalence Theorem, which states that a consistent, two level difference scheme is convergent if and only if it is stable. Most of the schemes are consistent. Therefore, the main difficulty is to prove stability. Intuitively speaking, a difference scheme is stable if small errors in the initial condition cause small errors in the solution.

Definition 3.2.1. (Stability according to Thomas (1995)).

A difference scheme is stable with respect to $\|\cdot\|_{l_2}$ if there exist $\Delta y_0, \Delta t_0 > 0$, $H, \beta \geq 0$ such that for all $\Delta y, \Delta \kappa \leq \Delta y_0$ and $\Delta t \leq \Delta t_0$

$$\|\mathbf{g}^{n+1}\|_{l_2} \leq H e^{\beta(n+1)\Delta t} \|\mathbf{g}^0\|_{l_2}.$$

For the following stability result, *Parseval's Identity* is essential. It says that the norm of \mathbf{g} in l_2 is equal to the norm of the *discrete Fourier transform* of \mathbf{g}

$$\hat{g}(\eta_1, \eta_2) = \frac{1}{2\pi} \sum_{i,j=-\infty}^{\infty} e^{-i\eta_1 i - i\eta_2 j} g_{i,j}$$

in the space of complex valued, Lebesgue square integrable functions on $[-\pi, \pi]^2$

$$L_2 := \left\{ \hat{g} : [-\pi, \pi]^2 \rightarrow \mathbb{C} : \int_{[-\pi, \pi]^2} |\hat{g}(\eta_1, \eta_2)|^2 d\eta_1 d\eta_2 < \infty \right\},$$

$$\|\hat{g}\|_{L_2} := \sqrt{\int_{[-\pi, \pi]^2} |\hat{g}(\eta_1, \eta_2)|^2 d\eta_1 d\eta_2}.$$

Define the *symbol of the difference scheme* p via

$$\hat{g}^{n+1} = p(\eta_1, \eta_2) \hat{g}^n.$$

Proposition 3.2.2. (Von Neumann condition according to Thomas (1995)).

The difference scheme for the initial-value problem is stable if and only if there exist $\Delta y_0, \Delta t_0, H > 0$ such that for all $\Delta y, \Delta \kappa \leq \Delta y_0, \Delta t \leq \Delta t_0$ and $\eta_1, \eta_2 \in [-\pi, \pi]$

$$|p(\eta_1, \eta_2)| \leq 1 + H\Delta t. \quad (3.44)$$

As Thomas (1995) points out, care must be taken when using the notion of stability from Definition 3.2.1. The von Neumann condition (3.44) with a nonzero H allows for exponential growth of the solution, i.e., Definition 3.2.1 only guarantees that the instability is less than or equal to an exponential. Let us calculate the symbol of our scheme (3.43). Define

$$r_y := \frac{\Delta t}{2\Delta y}, \quad r_{yy} := \frac{\Delta t}{(\Delta y)^2}, \quad r_\kappa := \frac{\Delta t}{2\Delta \kappa}, \quad r_{\kappa\kappa} := \frac{\Delta t}{(\Delta \kappa)^2}.$$

Then

$$\begin{aligned} g_{i,j}^{n+1} &= g_{i,j}^n (1 - \delta\Delta t + 2\delta_{\kappa\kappa}r_{\kappa\kappa}) + g_{i-1,j}^n \delta_y r_y - g_{i+1,j}^n \delta_y r_y \\ &+ g_{i,j-1}^n (\delta_\kappa r_\kappa - \delta_{\kappa\kappa}r_{\kappa\kappa}) - g_{i,j+1}^n (\delta_\kappa r_\kappa + \delta_{\kappa\kappa}r_{\kappa\kappa}). \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{g}^{n+1} &= \frac{1}{2\pi} \sum_{i,j=-\infty}^{\infty} e^{-i\eta_1 - ij\eta_2} g_{i,j}^{n+1} = \left\{ 1 - \delta\Delta t + 2\delta_{\kappa\kappa}r_{\kappa\kappa} + e^{-i\eta_1} \delta_y r_y - e^{i\eta_1} \delta_y r_y \right. \\ &+ \left. e^{-i\eta_2} (\delta_\kappa r_\kappa - \delta_{\kappa\kappa}r_{\kappa\kappa}) - e^{i\eta_2} (\delta_\kappa r_\kappa + \delta_{\kappa\kappa}r_{\kappa\kappa}) \right\} \hat{g}^n. \end{aligned} \quad (3.45)$$

That is

$$p(\eta_1, \eta_2) = [1 - \delta\Delta t + 2\delta_{\kappa\kappa}r_{\kappa\kappa} (1 - \cos(\eta_2))] - 2i [\delta_y r_y \sin(\eta_1) + \delta_\kappa r_\kappa \sin(\eta_2)].$$

Let $\delta_{\kappa\kappa} < 0$. If we choose r_{yy} and $r_{\kappa\kappa}$ as fixed constants with $r_{\kappa\kappa} < -\frac{1}{2\delta_{\kappa\kappa}}$, then there exists $\Delta t_0 > 0$ such that for all $\Delta t < \Delta t_0$

$$\begin{aligned} |p(\eta_1, \eta_2)|^2 &= [1 - \delta\Delta t + 2\delta_{\kappa\kappa}r_{\kappa\kappa} (1 - \cos(\eta_2))]^2 \\ &+ [\delta_y^2 r_{yy} + \delta_\kappa^2 r_{\kappa\kappa} + 2|\delta_y \delta_\kappa| \sqrt{r_{yy} r_{\kappa\kappa}}] \Delta t < 1. \end{aligned}$$

We can adapt this result to the finite difference scheme, that we introduced in Subsection 3.2.1. Since it is a backward instead of a forward scheme, $-\frac{1}{2}\sigma^2$ plays the role of $\delta_{\kappa\kappa}$. As experiments show, we get good numerical results by taking $r_{yy} = r_{\kappa\kappa} \approx \sigma_{max}^{-2} \wedge 1$ and Δt reasonably small. Notice that this stability condition, that we heuristically derived for the finite difference scheme, is similar to the positive transition probability condition (3.8) from the Markov chain method. In both cases, the square of the state space mesh size has to be of the order of the time step.

3.3 Cox-Ingersoll-Ross price impact

We discussed the numerical scheme based on the Markov chain method in Section 3.1 and based on the HJB variational inequality in Section 3.2. In Subsection 3.2.2, we pointed out the similarity of these two alternatives. Since we were able to prove convergence for the Markov chain method, we pick this one in its dimension-reduced form to exemplarily implement our optimization problem for the mean-reverting CIR process from *Assumption SpecialCIR* where

$$dK_s = \bar{\mu} (\bar{K} - K_s) ds + \bar{\sigma} \sqrt{K_s} dW_s^K.$$

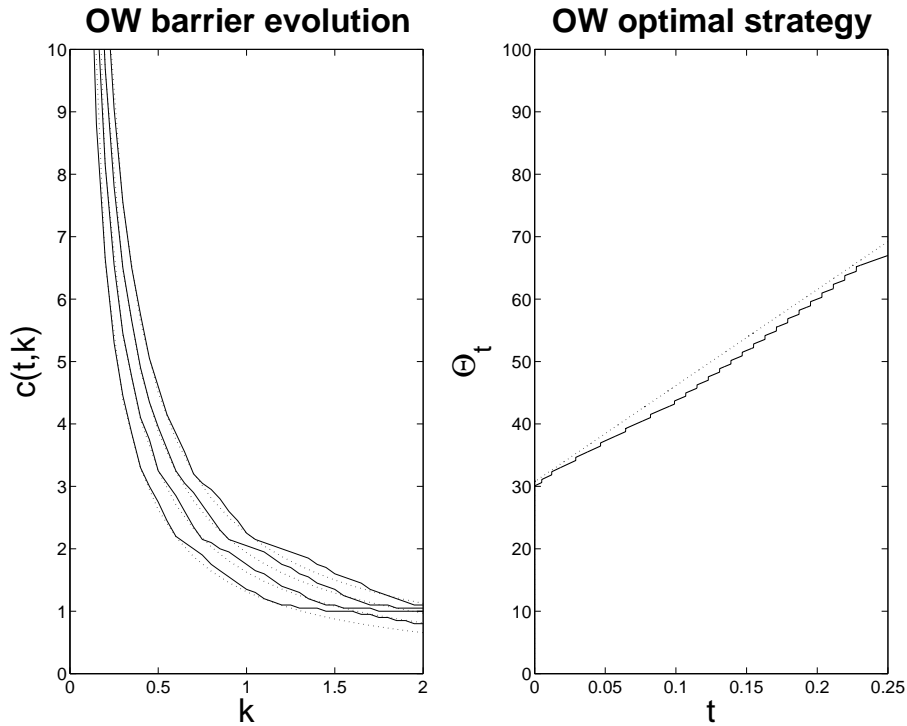


Figure 3.6: Numerical treatment of the constant liquidity case with $\rho = 5$, $T = 0.25$, $y_{max} = 30$, $\kappa_{max} = 2$, $\Delta t = 0.00005$, $h = 0.05$. *Left:* The numerically calculated and the theoretical (dashed) barrier between wait and buy region $c(t, \kappa)$ for several values of t ($t = 0, \frac{1}{4}T, \frac{1}{2}T, \frac{3}{4}T$ top down). *Right:* Comparison of the optimal strategy from our numerical scheme for $\delta = 0$, $x = 100$ with the theoretical value (dashed).

Specify nine parameters, in order to apply the numerical scheme (3.40)

$$\rho, \bar{\mu}, \bar{K}, \bar{\sigma}, \quad T, y_{max}, \kappa_{max}, \quad h, dt^h = \Delta t.$$

Without our earlier considerations, it would be hard to guess a good combination of these parameters such that the numerical scheme yields reasonable results. The transition probabilities of the Markov chain must be positive. Even in the constant

liquidity case with $\bar{\mu} = \bar{\sigma} = 0$, we have to take care of condition (3.9). An example is given in Figure 3.6. Our numerical results are close to the theoretical optimal strategy given in Proposition 2.2.2.

In Figure 3.7, we state the result of our implementation for the CIR process for two different values of mean-reversion speed $\bar{\mu}$. The corresponding parameter choices satisfy the assumptions of Proposition 2.4.13 such that WR-BR structure is guaranteed. The size of the wait region is decreasing in time in both cases of Figure 3.7. The barrier profile $c(t, \kappa)$ is rather flat in κ for low mean-reversion, but clearly increasing in κ for high mean-reversion. That is according to Proposition 2.3.7, our trades are *aggressive in the liquidity* compared to the passive in the liquidity trades that we get for the GBM. The aggressive in the liquidity behavior becomes also clear, when one looks at the reaction of the optimal strategy $\Theta_t(\omega)$ on the illiquidity process $K_t(\omega)$. At times of high K_t , it falls behind the constant liquidity strategy, while trading is accelerated, when K_t is low. Observe that there are flat stretches in the optimal strategy in the picture at the bottom on the right. During these time spans, the optimal ratio of outstanding shares over order book deviation is in the interior of the wait region.

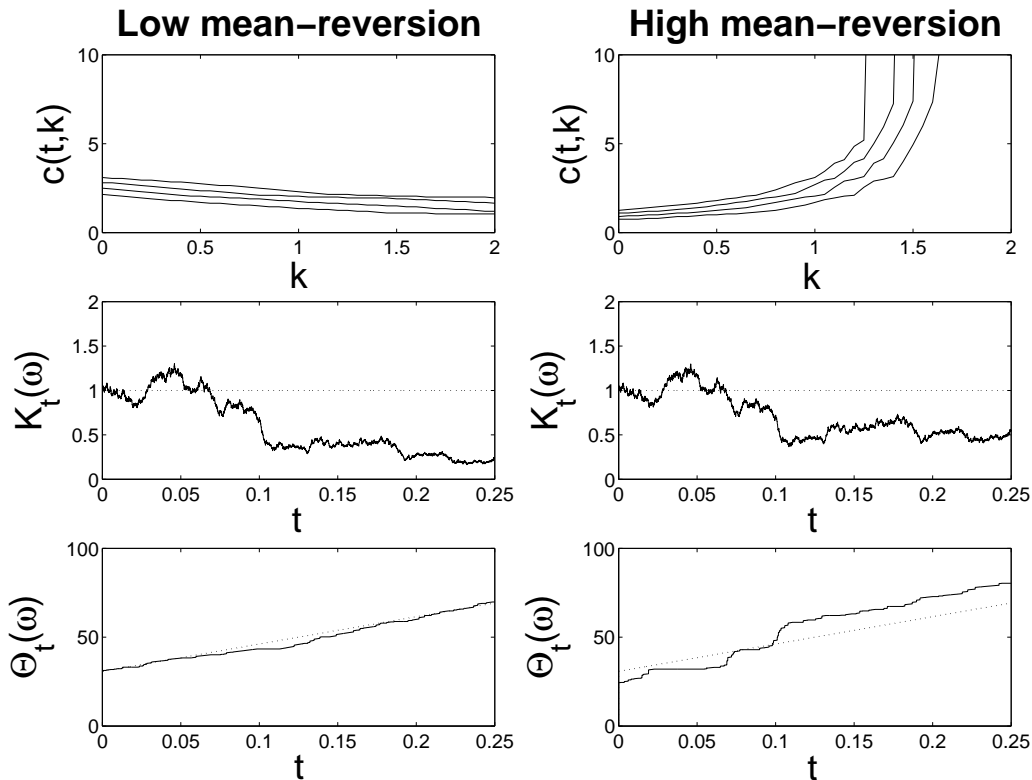


Figure 3.7: Implementation for the CIR process with $\rho = 5$, $\bar{K} = 1$, $\bar{\sigma}^2 = 2$, $T = 0.25$, $y_{max} = 30$, $\kappa_{max} = 2$, $\Delta t = 0.00005$, $h = 0.05$. *Left:* $\bar{\mu} = 3$. *Right:* $\bar{\mu} = 10$. A simple Euler scheme is used to simulate a scenario ω of the CIR process K . In the middle plots, $K_t(\omega)$ is compared to the mean-reverting level $\bar{K} = 1$. At the bottom, the optimal strategy for $\delta = 0$, $x = 100$ and ω is compared to the constant liquidity strategy (dashed).

Remark 3.3.1. (CIR: Situation at t_{N-1}).

As in Remark 2.3.4, we can assume discrete trading time and explicitly state the situation one instance before the end of the trading period. This yields the following barrier for $\tilde{a} := e^{-\rho(t_N-t_{N-1})}$, $\tilde{b} := e^{-\bar{\mu}(t_N-t_{N-1})}$

$$c^N(t_{N-1}, \kappa) = \left\{ \begin{array}{ll} \frac{1-\tilde{a}}{\bar{K}(1-\tilde{b})+\kappa(\tilde{b}-\tilde{a})} & \text{if } \bar{K}(1-\tilde{b}) + \kappa(\tilde{b}-\tilde{a}) > 0 \\ \infty & \text{otherwise} \end{array} \right\}. \quad (3.46)$$

Figure 3.8 illustrates the barrier $c^N(t_{N-1}, \cdot)$. It is increasing for $\bar{\mu} > \rho$ and decreasing for $\bar{\mu} < \rho$. This is consistent with our results from Figure 3.7 and analytically confirms the conjecture above, that the CIR process can lead to aggressive in the liquidity behavior. See more details in the following proposition.

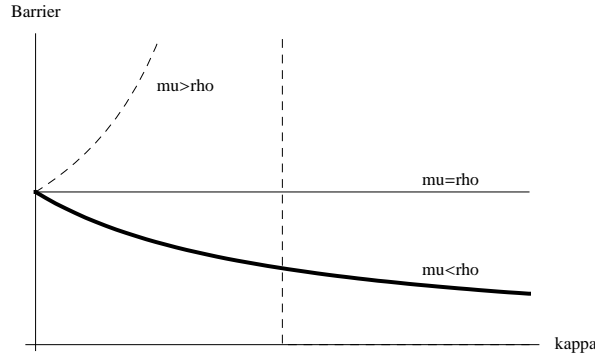


Figure 3.8: Schematic illustration of the barrier $c^N(t_{N-1}, \cdot)$ in the CIR case. We have $c^N(t_{N-1}, 0) = \frac{1-\tilde{a}}{\bar{K}(1-\tilde{b})}$ and in case of $\bar{\mu} > \rho$ the barrier has an asymptote at $\kappa = \bar{K} \frac{1-\tilde{b}}{\tilde{b}-\tilde{a}}$.

Proposition 3.3.2. (CIR: Aggressive/passive in the liquidity at t_{N-1} possible).

For $\bar{\mu} \geq \rho$, the trade $\xi_{N-1}^N(t_{N-1}, \delta, x, \kappa)$ is aggressive in the liquidity.

For $\bar{\mu} < \rho$, the trade is aggressive in the liquidity for $\frac{x}{\delta} \geq \frac{1-2\tilde{a}+\tilde{b}}{\bar{K}(1-\tilde{b})}$ and passive in the liquidity otherwise.

Proof. The first assertion follows from (3.46) and Proposition 2.3.7 a).

For $\bar{\mu} < \rho$, i.e., $\tilde{a} < \tilde{b}$, we get

$$\begin{aligned} c^N(t_{N-1}, \kappa) &= \frac{1-\tilde{a}}{\bar{K}(1-\tilde{b})+\kappa(\tilde{b}-\tilde{a})} < \frac{1-\tilde{a}}{\bar{K}(1-\tilde{b})} < \frac{1-2\tilde{a}+\tilde{b}}{\bar{K}(1-\tilde{b})}, \\ \xi_{N-1} &= \max \left\{ 0, \frac{x - c^N(t_{N-1}, \kappa) \delta}{1 + \kappa c^N(t_{N-1}, \kappa)} \right\}, \\ \frac{\partial}{\partial \kappa} \xi_{N-1} &= -\frac{(1-\tilde{a}) \left[(1-\tilde{b}) \bar{K} x - (1-2\tilde{a}+\tilde{b}) \delta \right]}{\left[(1-\tilde{b}) \bar{K} + (1-2\tilde{a}+\tilde{b}) \kappa \right]^2} \quad \text{for } \frac{x}{\delta} > c^N(t_{N-1}, \kappa). \end{aligned}$$

□

Numerical cost comparison

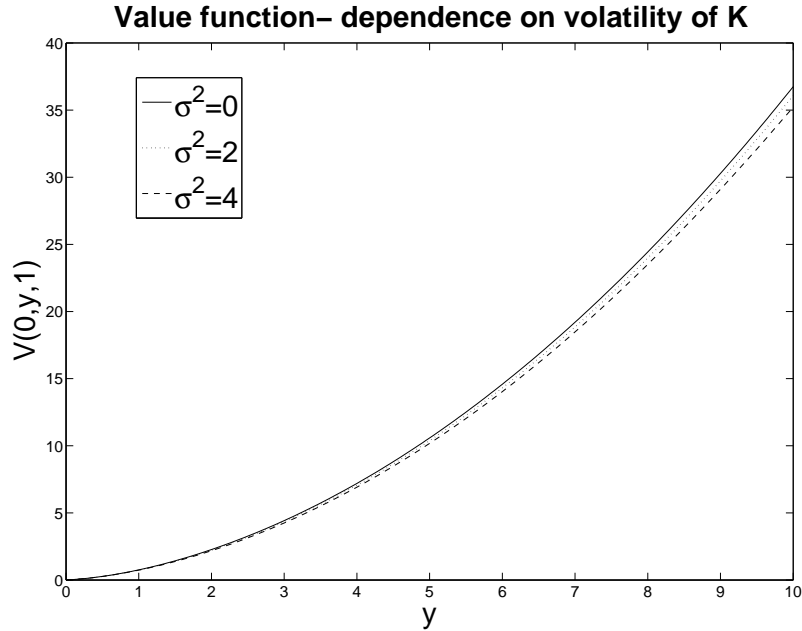


Figure 3.9: Value function at $t = 0$ and $\kappa = \bar{K} = 1$. The three lines correspond to $\bar{\sigma}^2 = 0$, 2 (dotted), 4 (dashed). All other parameters are chosen as in Figure 3.7 with $\bar{\mu} = 10$.

After having a look at the WR-BR barrier and the corresponding optimal strategy, let us now focus on the value function. Figure 3.9 compares markets that are assumed to have different fluctuations in the order book height. It turns out that the total expected costs are slightly decreasing in $\bar{\sigma}$, i.e., *it is beneficial to trade in markets with high impact volatility*. However, we cannot improve the visibility of this effect in the figure, since $\bar{\mu}\bar{K} > 2\bar{\sigma}^2$ from *Assumption SpecialCIR* restricts our choice to $\bar{\sigma}^2$ smaller than or equal to $\frac{\bar{\mu}\bar{K}}{2} = 5$. This problem also arises for other parameter choices, but the monotonicity of the value function in $\bar{\sigma}$ remains in all of these cases. This decrease of the value function in $\bar{\sigma}$ can heuristically be explained. The value function for constant liquidity is increasing and concave in κ , as we have seen in Proposition 2.2.1 and 2.2.2. For instance, let $0 < \check{\kappa} < \hat{\kappa}$ and compare the costs, when $K_s \equiv \frac{\check{\kappa} + \hat{\kappa}}{2}$ on $[t, T]$, to the expected costs, when $K_s \equiv \check{\kappa}$ or $\hat{\kappa}$ both with probability one half. Due to

$$U^{OW} \left(t, \delta, x, \frac{\check{\kappa} + \hat{\kappa}}{2} \right) > \frac{1}{2} [U^{OW}(t, \delta, x, \check{\kappa}) + U^{OW}(t, \delta, x, \hat{\kappa})],$$

one is better off in the case with randomness, which is meant to correspond to the high volatility case in Figure 3.9. We show in Lemma 3.3.5 that for $\delta = 0$, the value function is converging to zero as $\bar{\sigma} \rightarrow \infty$.

We have set up our optimization problem by minimizing the expectation of the costs. For the resulting optimal strategy, it is interesting to see what variance the costs have.

In Figure 3.10, we show the numerically calculated cumulative distribution function of the costs. We compare it to the cost distribution resulting from the numerically calculated constant liquidity strategy and the reoptimized constant liquidity strategy. By the reoptimized strategy, we mean the strategy that we get, when we calculate a strategy from the constant liquidity barrier

$$c(t, \kappa) = \frac{\rho(T - t) + 1}{\kappa}$$

using the actual, time-varying K_t . This corresponds to a trader, that believes in a constant order-book, but readjusts the constant, each time he sees a change of the order book height. His strategy will then be passive in the liquidity. Therefore, the corresponding cost distribution has even more mass at high cost values than the static constant liquidity strategy.

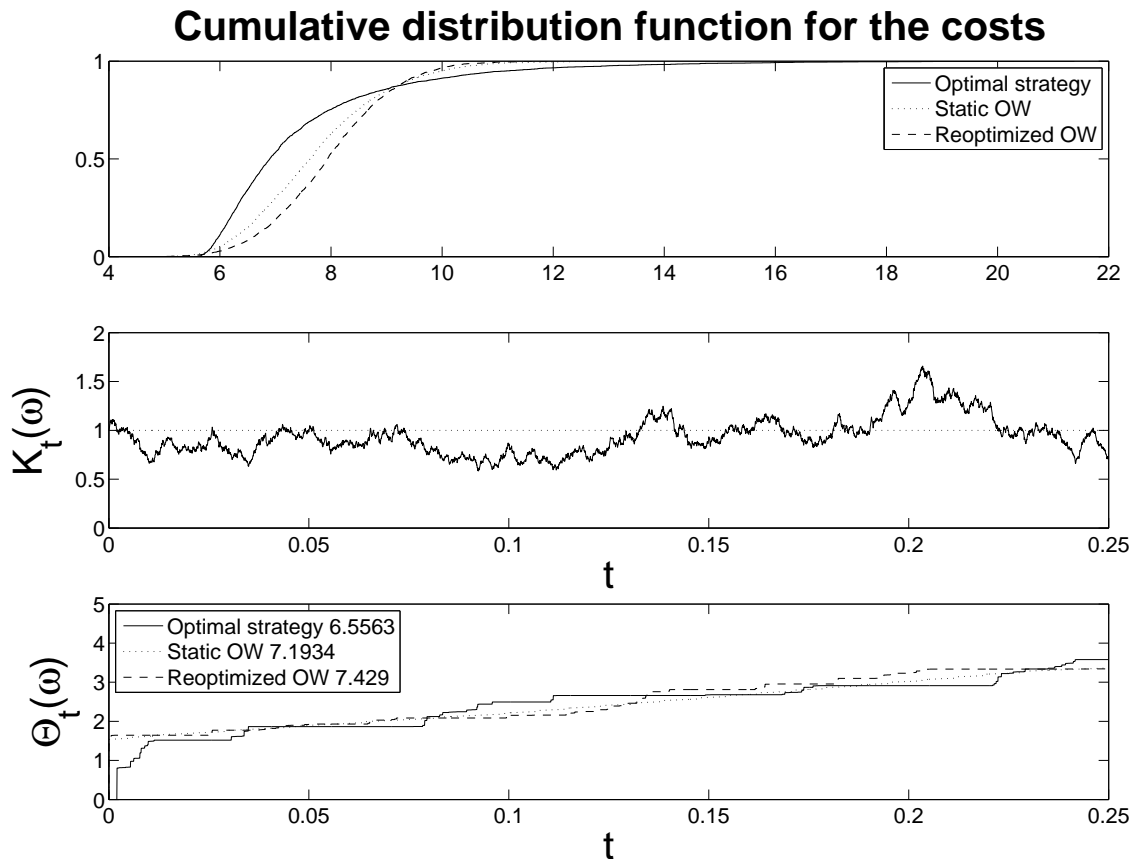


Figure 3.10: We have chosen $t = 0$, $\delta = 0$, $x = 5$, $K_0 = \bar{K}$, $\bar{\mu} = 10$, $\bar{\sigma}^2 = 4$. All other parameters are chosen as in Figure 3.7. *Top:* Illustration of the cumulative distribution function for the costs resulting from 3000 scenarios of the CIR process. *Middle:* One generic scenario for K . *Bottom:* The corresponding strategies. The numbers in the legend are the costs for the shown scenario using the corresponding strategy.

Analytical cost comparison

Consider a block-shaped limit order book market with K being a CIR process with volatility $\bar{\sigma}$ and constant resilience speed. We then want to compare the optimal expected costs $U^{\bar{\sigma}}(0, 0, x, \kappa)$ to the expected costs $J^{\bar{\sigma}}(\Theta^{OW})$, when a trader applies the optimal constant liquidity strategy. It turns out in Lemma 3.3.3, that we can explicitly calculate $J^{\bar{\sigma}}(\Theta^{OW})$, and that this term does not depend on the volatility. Afterwards, we will see in Lemma 3.3.4, that we can use the results from the Euler-Lagrange section to calculate $U^{\bar{\sigma}=0}(0, 0, x, \kappa)$ analytically. Since we have numerically examined in Figure 3.9, that $U^{\bar{\sigma}}$ is decreasing in $\bar{\sigma}$, we then know that the explicitly known term $(J^{\bar{\sigma}}(\Theta^{OW}) - U^{\bar{\sigma}=0})$ is a lower bound for the expected cost saving $(J^{\bar{\sigma}}(\Theta^{OW}) - U^{\bar{\sigma}})$.

Lemma 3.3.3. (CIR expected costs of constant liquidity strategy).

Let Assumption SpecialCIR hold. Denote by $\Theta^{OW} \in \mathcal{A}_0(x)$ the strategy given in (2.3). Then for all $\bar{\sigma} \in [0, \infty)$,

$$\begin{aligned} U^{\bar{\sigma}}(0, 0, x, \kappa) &\leq J^{\bar{\sigma}}(0, 0, \Theta^{OW}, \kappa) \\ &= \frac{\kappa(3\bar{\mu} + 2\rho) + \bar{K}(\bar{\mu} - 2\rho + 2\bar{\mu}\rho T) + (\kappa - \bar{K})(\bar{\mu} - 2\rho)e^{-\bar{\mu}T}}{2\bar{\mu}(\rho T + 2)^2} x^2. \end{aligned} \quad (3.47)$$

In particular,

$$U^{\bar{\sigma}}(0, 0, x, \bar{K}) \leq U^{OW}(0, 0, x, \bar{K}). \quad (3.48)$$

Proof. Since Θ^{OW} is deterministic, we get

$$\begin{aligned} J^{\bar{\sigma}}(0, 0, \Theta^{OW}, \kappa) &= \frac{\kappa}{2} (\Delta\Theta_0^{OW})^2 + \int_{[0, T]} \mathbb{E}[D_s^{OW}] d\Theta_s^{OW} \\ &\quad + \left(\mathbb{E}[D_T^{OW}] + \frac{\mathbb{E}[K_T]}{2} \Delta\Theta_T^{OW} \right) \Delta\Theta_T^{OW}. \end{aligned} \quad (3.49)$$

Hence, we need to calculate the expected price deviation, which results from Θ^{OW} . For $s \in (0, T]$, dynamic (1.7) yields

$$\mathbb{E}[D_s^{OW}] = \kappa \Delta\Theta_0^{OW} + \int_{[0, s)} \mathbb{E}[K_u] d\Theta_u^{OW} - \rho \int_{[0, s)} \mathbb{E}[D_u^{OW}] du.$$

That is $\mathbb{E}[D_s^{OW}]$ satisfies the initial value ordinary differential equation

$$\frac{\partial}{\partial s} \mathbb{E}[D_s^{OW}] = (\bar{K} + (\kappa - \bar{K})e^{-\bar{\mu}s}) \frac{\rho x}{\rho T + 2} - \rho \mathbb{E}[D_s^{OW}], \quad \mathbb{E}[D_{0+}^{OW}] = \kappa \Delta\Theta_0^{OW}.$$

It is solved by

$$\mathbb{E}[D_s^{OW}] = \frac{(\kappa - \bar{K})(\bar{\mu}e^{-\rho s} - \rho e^{-\bar{\mu}s}) + \bar{K}(\bar{\mu} - \rho)}{(\bar{\mu} - \rho)(\rho T + 2)} x.$$

Plugging this into equation (3.49) yields the assertion. Inequality (3.48) follows from Proposition 2.2.2 and by setting $\kappa = \bar{K}$ in (3.47). \square

Lemma 3.3.4. (CIR value function for zero volatility).

Let Assumption *SpecialCIR* with $\bar{\sigma} = 0$ hold. Then

$$U^{\bar{\sigma}=0}(0, 0, x, \kappa) \geq \frac{\kappa \bar{K} \bar{\mu}}{2\bar{K} \bar{\mu} + \kappa \rho \log \left(\frac{(\kappa - \bar{K})(\bar{\mu} - 2\rho) - 2e^{\bar{\mu}T} \bar{K} \rho}{\bar{\mu}(\kappa - \bar{K}) - 2\kappa \rho} \right)} x^2.$$

Equality applies for each of the following cases:

1. $\bar{\mu} \in (0, \rho] \cup \{2\rho\}$
2. $\bar{\mu} \in (\rho, 2\rho)$ and $\kappa \leq \bar{K} \left(1 + 2 \frac{\rho^2}{\bar{\mu}(\bar{\mu} - 3\rho)} \right)^{-1}$

Proof. Recall Subsection 2.2.4, where we discussed closed form solutions of our problem for deterministic K . Due to Assumption *SpecialCIR*, $2\rho \geq \bar{\mu} > 0$. Hence, we can apply Lemma 2.2.24 and thus

$$U^{\bar{\sigma}=0}(0, 0, x, \kappa) \geq \int_{[0, T]} D_s \frac{D'_s + \rho D_s}{K_s} ds + \frac{D_{0+}^2}{2K_0} + \frac{D_{T+}^2 - D_T^2}{2K_T} \quad (3.50)$$

with $D_s = D_{T+} f_s$, $f_s = \frac{K'_s + \rho K_s}{K'_s + 2\rho K_s}$ for $s \in (0, T]$ and

$$D_{T+} = \left(\int_{[0, T]} \frac{f'_s + \rho f_s}{K_s} ds + \frac{f_0}{\kappa} + \frac{1 - f_T}{K_T} \right)^{-1} x.$$

Plugging all this into the right-hand side of (3.50) yields the desired inequality.

The additional assumptions in the lemma assure that the condition

$$f'_s + \rho f_s = \rho \frac{\kappa + (e^{\bar{\mu}s} - 1) \bar{K}}{((\kappa - \bar{K})(\bar{\mu} - 2\rho) - 2e^{\bar{\mu}s} \rho \bar{K})^2} \{ 2e^{\bar{\mu}s} \rho^2 \bar{K} + (\kappa - \bar{K})(\bar{\mu} - \rho)(\bar{\mu} - 2\rho) \} \geq 0$$

is met on $[0, T]$. Hence, Proposition 2.2.22 applies and this guarantees equality in (3.50). \square

Lemma 3.3.5. (CIR value function, asymptotics for high volatility).

Let Assumption *SpecialCIR* hold, but neglect the restriction $\bar{\mu} \bar{K} > 2\bar{\sigma}^2$. Then

$$\delta e^{-\rho(T-t)} x \leq \lim_{\bar{\sigma} \rightarrow \infty} U^{\bar{\sigma}}(t, \delta, x, \kappa) \leq \delta x.$$

In particular,

$$\lim_{\bar{\sigma} \rightarrow \infty} U^{\bar{\sigma}}(t, 0, x, \kappa) = 0.$$

Proof. Fix $\epsilon > 0$ as well as (t, δ, x, κ) and define

$$\tau_\epsilon := \inf \{ s \in [t, \infty) \mid K_s \leq \epsilon \}.$$

Choose $\Theta^\epsilon \in \mathcal{A}_t(x)$ to be the strategy that unwinds the whole order at $\min(\tau_\epsilon, T)$, i.e. for $s \in [t, T]$

$$\Theta_s^\epsilon := x \mathbb{I}_{(\tau_\epsilon \wedge T, T]}(s), \quad \Theta_{T+}^\epsilon := x.$$

Then

$$\begin{aligned} U^{\bar{\sigma}}(t, \delta, x, \kappa) &\leq J^{\bar{\sigma}}(t, \delta, \Theta^\epsilon, \kappa) \\ &\leq \mathbb{E} \left[\mathbb{I}_{\{\tau_\epsilon \leq T\}} \left(\delta + \frac{\epsilon}{2} x \right) x + \mathbb{I}_{\{\tau_\epsilon > T\}} \left(\delta + \frac{K_T}{2} x \right) x \right] \\ &\leq \left(\delta + \frac{\epsilon}{2} x \right) x + \mathbb{P}[\tau_\epsilon > T] \delta + \mathbb{E} [K_T \mathbb{I}_{\{\tau_\epsilon > T\}}] \frac{x^2}{2}. \end{aligned}$$

In the following, we argue that the right-hand side converges to $(\delta + \frac{\epsilon}{2} x)x$ as $\bar{\sigma} \rightarrow \infty$. First of all, use Markov's inequality and consider

$$\mathbb{P}[\tau_\epsilon > T] = \mathbb{P} \left[\inf_{s \in [t, T]} K_s > \epsilon \right] \leq \frac{\mathbb{E} [\inf_{s \in [t, T]} K_s]}{\epsilon}.$$

Jensen's inequality yields

$$\mathbb{E} \left[\inf_{s \in [t, T]} K_s \right] \leq \left(\mathbb{E} \left[\sup_{s \in [t, T]} q_s \right] \right)^{-1}.$$

Together with our considerations from (2.39),

$$\lim_{\bar{\sigma} \rightarrow \infty} \mathbb{P}[\tau_\epsilon > T] = 0.$$

With dominated convergence, it also follows that

$$\lim_{\bar{\sigma} \rightarrow \infty} \mathbb{E} [K_T \mathbb{I}_{\{\tau_\epsilon > T\}}] = 0.$$

□

This result also holds for more general processes K that get arbitrary small as their volatility gets large, in the sense that for all $\epsilon > 0$

$$\mathbb{P} \left[\inf_{s \in [t, T]} K_s > \epsilon \right]$$

converges to zero as the volatility of K approaches infinity.

Chapter 4

Stochastic resilience

Kyle (1985) distinguishes three aspects of liquidity in a market: Spread, market depth and resilience. When liquidating a portfolio, we trade in one direction as argued in Proposition 1.1.2. Therefore, we can ignore the spread in our modeling. So far, we have analyzed the situation when the order book height, which corresponds to market depth, is stochastic. For simplicity, we assumed the resilience speed ρ to be deterministic. Let us now do it the other way around and thus do a first step towards a stochastic resilience model. Thereby, we keep the market depth model simple by taking a linear price impact with constant coefficient κ . But the crucial difference between modeling κ stochastically versus ρ is, that the resilience speed is not observable, whereas one can directly see the order book height in the market. Using the language of stochastic filtering, this means that the signal, which is the resilience speed, cannot be measured directly, but the best ask price $A_t = A_t^u + D_t$ can be used as an observation process to have a partial measurement of the signal. One might wish to model stochastic resilience by keeping the dynamics of D with a fixed constant ρ as before, but adding a noise term. Then the difficulty arises, that we cannot observe the noise of A^u and D separately, since we only monitor A in the market. Notice also that the observation process A depends on the strategy of the large investor, which makes the whole consideration rather complex.

It turns out that even the following simple model is astonishingly hard to treat. We take the impact coefficient κ to be a positive constant. Call the total number of shares to be traded $x > 0$. Furthermore, let ρ be a random variable, which is not time dependent and only takes two values l, h with $0 < l < h$ (low, high). We guess an a priori distribution of ρ with $\mathbb{P}[\rho = h] = r_0$ with a constant $r_0 \in (0, 1)$. For instance, take r_0 from historical data. Trading is allowed at three trading times $t_0 = 0, t_1 = 1$ and $t_2 = 2$. The evolution of the observable best ask price is assumed to be

$$\begin{aligned} A_0 &= A_0^u, \\ A_1 &= A_1^u + \kappa \xi_0 e^{-\rho} = (A_0^u + \Delta A_1^u) + \kappa \xi_0 e^{-\rho}, \\ A_2 &= A_2^u + \kappa \xi_0 e^{-2\rho} + \kappa \xi_1 e^{-\rho} = (A_0^u + \Delta A_1^u + \Delta A_2^u) + \kappa \xi_0 e^{-2\rho} + \kappa \xi_1 e^{-\rho}. \end{aligned} \quad (4.1)$$

The increments of the unaffected best ask price are independent and normally distributed with zero mean and variance σ^2 . At time t_1 , we can observe A_1 , but not ΔA_1^u and $e^{-\rho}$ individually. Nevertheless, we can use our information of A_1 at t_1 to update our beliefs about the distribution of ρ . Using the normal density $\frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{z^2}{2\sigma^2})$ yields

$$\begin{aligned} r_1 &:= \mathbb{P}[\rho = h | \mathcal{F}_1] \\ &= \frac{r_0 \mathbb{P}[\Delta A_1^u = A_1 - A_0 - \kappa \xi_0 e^{-h}]}{r_0 \mathbb{P}[\Delta A_1^u = A_1 - A_0 - \kappa \xi_0 e^{-h}] + (1 - r_0) \mathbb{P}[\Delta A_1^u = A_1 - A_0 - \kappa \xi_0 e^{-l}]} \\ &= \frac{r_0 \exp\left(-\frac{1}{2\sigma^2} [A_1 - A_0 - \kappa \xi_0 e^{-h}]^2\right)}{r_0 \exp\left(-\frac{1}{2\sigma^2} [A_1 - A_0 - \kappa \xi_0 e^{-h}]^2\right) + (1 - r_0) \exp\left(-\frac{1}{2\sigma^2} [A_1 - A_0 - \kappa \xi_0 e^{-l}]^2\right)}. \end{aligned}$$

That is the conditional probability of a high resilience speed given A_1 is a function of the initial trade ξ_0 , A_1 , σ^2 and κ .

With this model in mind, we now proceed by a backward induction to compute the optimal trades ξ_n^* for $n = 0, 1, 2$. The aim is to then examine the dependence of the optimal trade ξ_n^* on the observation A_n . In case of a high observation of A_n , which suggests ρ to be low, do we optimally trade more or less? This is the question we want to answer. If ξ_n^* is decreasing (increasing) in A_n , we say that our trade is *aggressive in the money* (*passive in the money*).

Proposition 4.0.1. (Aggressive in the money).

For arbitrary $\xi_0 \in [0, x]$, $\xi_2^* = x - \xi_0 - \xi_1^*$ with

$$\xi_1^*(\xi_0, A_1, \sigma^2, \kappa) = \frac{1}{2} \left(x - \xi_0 \frac{r_1(\xi_0, A_1, \sigma^2, \kappa) e^{-2h} + (1 - r_1(\xi_0, A_1, \sigma^2, \kappa)) e^{-2l} - 1}{r_1(\xi_0, A_1, \sigma^2, \kappa) e^{-h} + (1 - r_1(\xi_0, A_1, \sigma^2, \kappa)) e^{-l} - 1} \right).$$

Since ξ_1^* is decreasing in A_1 , this corresponds to a trade which is *aggressive in the money*.

Proof. For a given initial trade ξ_0 , minimize the expected cost from trading ξ_1 shares at t_1 and $x - \xi_0 - \xi_1$ shares at t_2 with respect to ξ_1 , i.e., consider

$$J(\xi_1) := \mathbb{E} \left[\xi_1 \left(A_1 + \frac{\kappa}{2} \xi_1 \right) + (x - \xi_0 - \xi_1) \left(A_2 + \frac{\kappa}{2} (x - \xi_0 - \xi_1) \right) \middle| \mathcal{F}_1 \right].$$

Except of A_2 , all terms are \mathcal{F}_1 -measurable. From (4.1),

$$\mathbb{E}[A_2 | \mathcal{F}_1] = A_1 + r_1 \kappa [\xi_0 (e^{-2h} - e^{-h}) + \xi_1 e^{-h}] + (1 - r_1) \kappa [\xi_0 (e^{-2l} - e^{-l}) + \xi_1 e^{-l}].$$

Plugging this into J , we get ξ_1^* by a straightforward calculation. Hence,

$$\begin{aligned} \frac{\partial}{\partial A_1} \xi_1^* &= - \frac{\frac{r_0}{2\sigma^2} \xi_0^2 (1 - r_0) (e^h - 1) (e^l - 1) (e^h - e^l)^2 e^{-(h+l)}}{2e^{h+l} (e^h - 1) (e^l - 1) (1 - r_0) r_0 + \tilde{f}(h, l) (1 - r_0)^2 + \tilde{f}(l, h) r_0^2} < 0 \text{ with} \\ \tilde{f}(a, b) &:= \exp \left\{ 2a + \frac{(A_0 - A_1 + \kappa \xi_0 e^{-a})^2 - (A_0 - A_1 + \kappa \xi_0 e^{-b})^2}{2\sigma^2} \right\} (e^b - 1)^2. \end{aligned}$$

□

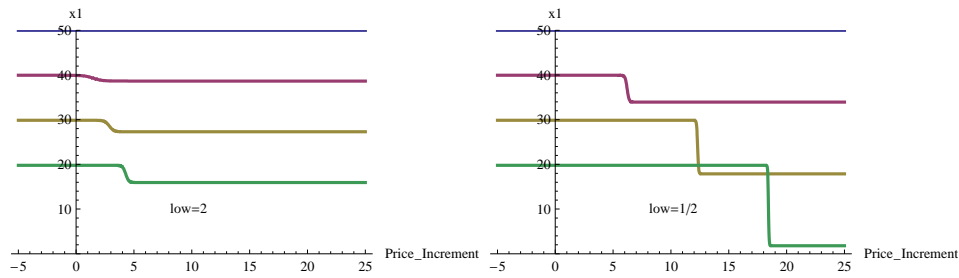


Figure 4.1: Dependence of ξ_1^* on the observable price increment $A_1 - A_0$ for $x = 100$, $r_0 = \frac{1}{2}$, $\kappa = 1$, $\sigma^2 = 1$, $h = 5$ and $\xi_0 = 0, 20, 40, 60$ top-down. *Left:* $l = 2$. *Right:* $l = \frac{1}{2}$.

Figure 4.1 illustrates the statement of Proposition 4.0.1. The main aspect to notice is that the optimal strategy is stochastic, in the sense that it reacts to the price movement $A_1 - A_0$. More precisely, ξ_1^* is aggressive in the money. This effect is more pronounced the higher we choose ξ_0 . It is not obvious, how to get the optimal trade ξ_0^* in closed form. Therefore, we discuss the properties of ξ_1^* for arbitrary values of $\xi_0 \in [0, x]$.

We have $\xi_1^*(0, A_1, \sigma^2, \kappa) \equiv \frac{1}{2}x$, which is intuitively clear, since A_1 does not give any information about ρ if we have not traded at t_0 . At the same time, if the variance of the unaffected best ask price σ^2 rises, a small movement $A_1 - A_0$ in price is less informative with respect to the resilience. Therefore, it takes higher amplitudes of $A_1 - A_0$ to see significant differences in ξ_1^* . This is illustrated in Figure 4.2.

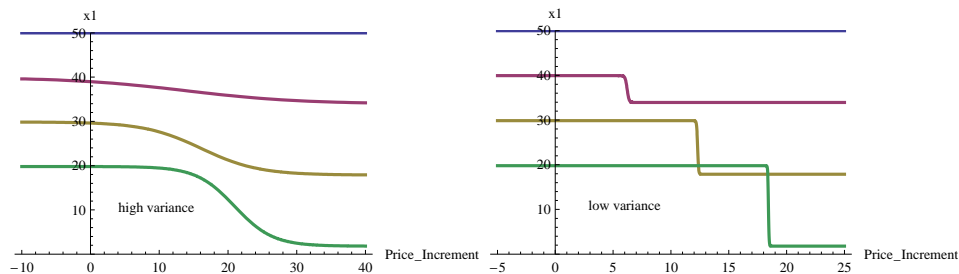


Figure 4.2: Dependence of ξ_1^* on the observable price increment $A_1 - A_0$ for parameters as in Figure 4.1 with $l = \frac{1}{2}$. *Left:* $\sigma^2 = 100$. *Right:* $\sigma^2 = 1$.

Since $\lim_{A_1 \rightarrow \infty} r_1(\xi_0, A_1, \sigma^2, \kappa) = 0$ and $\lim_{A_1 \rightarrow -\infty} r_1(\xi_0, A_1, \sigma^2, \kappa) = 1$ respectively,

$$\Delta \xi_1^* := \lim_{A_1 \rightarrow -\infty} \xi_1^*(\xi_0, A_1, \sigma^2, \kappa) - \lim_{A_1 \rightarrow \infty} \xi_1^*(\xi_0, A_1, \sigma^2, \kappa) = \frac{1}{2} \xi_0 \left(\frac{e^{-2l} - 1}{e^{-l} - 1} - \frac{e^{-2h} - 1}{e^{-h} - 1} \right).$$

Note that the function

$$g(z) := \frac{e^{-2z} - 1}{e^{-z} - 1}$$

is strictly decreasing on $[0, \infty)$ with $\lim_{z \rightarrow 0} g(z) = 2$ as well as $\lim_{z \rightarrow \infty} g(z) = 1$. This leads to $\Delta \xi_1^*$ to be positive and smaller than $\frac{1}{2} \xi_0$. The function g is particularly steep for small values of z . Therefore, the trade ξ_1^* is particularly sensitive with respect to A_1 (measured by $\Delta \xi_1^*$) in situations, where the estimations h and l for the resilience lie far apart from each other and l is quite low.

Finally, imagine a trader who speculates if and how his trades should react to changes in the price because they might tell him something about the actual resilience. Then our model suggests, that he actually should react to these price changes. Namely, he should behave aggressive in the money. How quantitatively pronounced this behavior should be, depends highly on the estimation of the resilience (the values h and l) done by the trader.

As we have seen, the optimal execution problem with stochastic resilience is interesting from a practical point of view and at the same time, it is mathematically challenging due to the involved filtering. Therefore, an expanded model with more than three trading instances seems to be a promising topic for future research.

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