Optimal trade execution in order books with stochastic liquidity

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Abstract

In financial markets, liquidity changes randomly over time. We consider such random variations of the depth of the order book and evaluate their influence on optimal trade execution strategies. If the stochastic structure of liquidity changes satisfies certain conditions, then the unique optimal trading strategy exhibits a conventional structure with a single wait region and a single buy region and profitable round trip strategies cannot exist. In other cases, optimal strategies can feature multiple wait regions and optimal trade sizes that can be decreasing in the size of the position to be liquidated. Furthermore round trip strategies can be profitable depending on bid-offer spread assumptions. We illustrate our findings with several examples including the CIR model for the evolution of liquidity.

KEYWORDS: Market impact model, optimal order execution, limit order book, resilience, time-varying liquidity, profitable round trip trading strategies

1 Introduction

Liquidity is not constant throughout the day, but instead varies over time. Traders active in a market are typically expected to continuously observe these changes in liquidity and adjust their trades accordingly. Some part of the liquidity changes is driven by deterministic changes in expected liquidity levels, e.g., daily and weekly patterns as well as expected changes around important points in time such as news releases. These expected changes however do not explain liquidity variation fully. An unpredicted component of liquidity changes remains which can dominate the deterministic component.

We extend existing limit order book models and introduce a stochastic depth of the order book. In this market, we consider an investor who wants to purchase a large asset position. If the order book dynamics are driven by a general diffusion satisfying certain conditions, then we prove existence and uniqueness of the optimal trade execution strategy. This trading strategy exhibits a wait region / buy region structure with a single wait region and a single buy region. If the investor finds herself in the wait region at a given point in time, then she does not place any orders at this point; if she is in the buy region, then the investor buys just enough to bring her position from within the buy region to the boundary of the wait region.

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If limit order book depth is not driven by a diffusion satisfying said conditions, then the classical wait region / buy region structure with one region each does not need to hold. While optimal strategies may still be described in terms of wait and buy regions, there can be more than one of these regions. We provide several examples with such non-standard optimal trading strategies. Intuitively expected features do not need to hold any more. For example, the trade size at a given point in time can vary non-monotonically with the size of the remaining position: if a large or small position remains, then no order is placed, however a purchase order is placed if the remaining position is of medium size. To the best of our knowledge, a nonintuitive structure of such type in solutions of Markovian control problems was never observed in the literature.

The condition ensuring wait region / buy region structure also guarantees that round trip trading strategies cannot be profitable. If the condition is violated, then round trip strategies can generate profits if the bid-offer spread is assumed to be zero; if a dynamic spread is assumed, then profits from round trip strategies remain unavailable.

The majority of the optimal trade execution literature considers one of two different market models. First, several models assume an instantaneous temporary price impact, e.g., Almgren and Chriss (2001) and Almgren (2003). In these models, the temporary price impact at time t is independent of all orders executed at time prior to t and does not influence any order at a time after t, which greatly simplifies the analysis. Cheridito and Sepin (2014) and Almgren (2012) have studied stochastic temporary price impact in this setting and provide numerical methods for calculation of the optimal strategy and value function. In a second group of models, inspired by a limit order book interpretation, resilience is finite and depth and resilience are separately modelled. Our model falls into this second group. Due to the finite resilience of the order book, the execution price at time t is influenced by orders filled at times prior to t, and the execution at time t in turn influences the execution price of subsequent orders. Most of the existing literature assumes the liquidity parameters to be constant over time, see, e.g., Bouchaud, Gefen, Potters, and Wyart (2004), Obizhaeva and Wang (2013), Alfonsi, Fruth, and Schied (2010) and Predoiu, Shaikhet, and Shreve (2011). Alfonsi and Acevedo (2014), Bank and Fruth (2014) and Fruth, Schöneborn, and Urusov (2014) allow for deterministic changes in liquidity and are therefore closely related to our paper. Let us, however, point out that this paper is qualitatively different from the aforementioned papers, and the main differences are as follows. Due to the stochasticity in the depth of the order book, the optimal execution strategies in the framework of this paper are no longer deterministic (the latter was the case in the aforementioned group of papers). More surprisingly, the counterexamples to the wait region / buy region structure mentioned above appear in the framework of our present paper only. To the best of our knowledge, Chen, Kou, and Wang (2015) is the only paper considering stochastically varying limit order book depth. They provide a numerical method for calculation of the optimal strategy and value function in discrete time with the depth of the limit order book driven by a discrete Markov chain. In contrast, we focus on analytical results in a continuous time setting with limit order book depth following a diffusive process.

Starting with Huberman and Stanzl (2004), profitable round trip strategies haven been studied in a variety of market models by Gatheral (2010), Alfonsi, Schied, and Slynko (2012) and Klöck, Schied, and Sun (2014) among others. To the best of our knowledge, all existing literature on this topic assumes deterministic liquidity.

The remainder of this paper is structured as follows. In Section 2, we introduce a limit order book model with stochastic depth and derive basic structural features in Section 3. We prove existence and uniqueness of optimal strategies as well as the wait region / buy region structure in Section 4 as long as the stochastic dynamics of the limit order book depth obeys certain conditions. We apply these results to several examples of diffusive processes in Section 5. If the conditions of Section 4 are violated, then the optimal strategy does not need to be of wait region / buy region structure any more as we demonstrate in several examples in Section 6. In Section 7, we extend our model to two sided limit order books and investigate the returns of round trip trading strategies. We conclude in Section 8.

2 Model description

A limit order book model with time dependent depth was introduced in Fruth, Schöneborn, and Urusov (2014). In this previous paper we explain the model in depth and provide an economic motivation. In the following, we recapitulate the central components and notation and extend the model from deterministic order book depth to stochastic order book depth.

The model is built on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\in[0,T]}, \mathbb{P})$. As usual in dynamic programming we consider a general initial time $t \in [0,T]$ below. For the evolution of the trader's asset position over time interval $[t,T]$, we consider the set of *admissible* continuous-time increasing strategies

$$\mathcal{A}_t^{cts} := \big\{\Theta\colon \Omega \times [t, T+] \to [0,\infty) \mid \\ (\mathcal{F}_s) - \text{adapted, increasing, bounded, càglàd with } \Theta_t = 0\big\}$$

and denote $\xi_s := \Delta\Theta_s := \Theta_{s+} - \Theta_s$. In particular, absolutely continuous trading as well as impulse trades are allowed. A strategy from $\mathcal{A}_t^{cts}$ consists of a left-continuous process $(\Theta_s)_{s\in[t,T]}$ and an additional random variable Θ_{T+} with $\Delta\Theta_T = \Theta_{T+} - \Theta_T \geq 0$ being the last trade of the strategy. Let us emphasize that admissible strategies are bounded by definition, that is, for $\Theta \in \mathcal{A}_t^{cts}$, we have $\Theta_{T+} \leq \text{const} < \infty$ a.s. (the constant depends on a strategy). Denote by

$$\mathcal{A}_t^{cts}(x) := \big\{\Theta \in \mathcal{A}_t^{cts} \mid \Theta_{T+} = x \text{ a.s.}\big\} \tag{1}$$

the admissible strategies that build up a position of $x \in [0,\infty)$ shares until time T almost surely. For the majority of this paper, we consider only one side of the limit order book (namely, the buy side) and hence only include increasing strategies in $\mathcal{A}_t^{cts}$. As we will see in Section 7, selling cannot reduce overall purchase costs if the bid-offer spread is influenced by the trader.

In addition to continuous time, we will also consider trading in discrete time, i.e., at times

$$0 = t_0 < t_1 < ... < t_N = T.$$

In this case, we constrain our admissible strategy set to

$$\mathcal{A}_t^{dis} := \big\{\Theta \in \mathcal{A}_t^{cts} \mid \Theta_s = 0 \text{ on } [t, t_{\tilde{n}(t)}] \text{ and} \\ \Theta_s = \Theta_{t_n+} \text{ a.s. on } (t_n, t_{n+1}) \text{ for } n = \tilde{n}(t), ..., N-1\big\} \subset \mathcal{A}_t^{cts}$$

with $\tilde{n}(t) := \inf\{n = 0, ..., N \mid t_n \geq t\}$ and define

$$\mathcal{A}_t^{dis}(x) := \big\{\Theta \in \mathcal{A}_t^{dis} \mid \Theta_{T+} = x \text{ a.s.}\big\}$$

as the discrete analogue to $\mathcal{A}_t^{cts}(x)$.

Let D be the price impact process, i.e. the deviation of the current ask price from its steady state level, K the illiquidity process, and ρ the (time-varying) resilience speed.

Standing Assumption.
(i) K is a (possibly time-inhomogeneous) $(\mathcal{F}_s)$-Markov process with state space $(0,\infty)$ and finite first moments.
(ii) $\rho\colon [0,T] \to (0,\infty)$ is a strictly positive Lebesgue-integrable deterministic function.

The deviation D_s results from past trades on $[t,s)$ in the following way

$$dD_s = -\rho_s D_s ds + K_s d\Theta_s, \quad D_t = \delta. \tag{2}$$

That is, for $s \in [t,T]$,

$$D_s = \int_{[t,s)} K_u e^{-\int_u^s \rho_r dr} d\Theta_u + \delta e^{-\int_t^s \rho_u du} \tag{3}$$

and, taking into account the last trade $\Delta\Theta_T$,

$$D_{T+} = \int_{[t,T]} K_u e^{-\int_u^T \rho_r dr} d\Theta_u + \delta e^{-\int_t^T \rho_u du}. \tag{4}$$

The process K describes the externally given dynamics of the order book depth $q := 1/K$, while D represents the movement of the order book block due to the trades of the large investor and the resilience effect.

For any fixed $t \in [0,T]$, $\delta \geq 0$ and $\kappa > 0$, we define the *cost function* $J(t,\delta,\cdot,\kappa)\colon \mathcal{A}_t^{cts} \to [0,\infty]$ as[1]

$$J(\Theta) := J(t,\delta,\Theta,\kappa) := \mathbb{E}_{t,\delta,\kappa}\left[\int_{[t,T]} \left(D_s + \frac{K_s}{2}\Delta\Theta_s\right) d\Theta_s\right], \tag{5}$$

i.e., the expected liquidity cost on the time interval $[t,T]$ when $D_t = \delta$ and $K_t = \kappa$. While we do not exclude the possibility of an infinite cost of a strategy $\Theta \in \mathcal{A}_t^{cts}$, it is worth noting that, for any $\Theta \in \mathcal{A}_t^{dis}$, the cost is finite due to our standing assumption. Starting with (5) we meet the following notational convention, which will be used throughout the paper: $\mathbb{P}_{t,\kappa}$ is the probability measure under which the Markov process K starts at time t from κ, $\mathbb{E}_{t,\kappa}$ is the expectation under $\mathbb{P}_{t,\kappa}$, and we write $\mathbb{E}_{t,\delta,\kappa}$ for the expectation when the expression contains the process D and the starting point at time t in (2) is δ.

Let us now define our *value function for continuous trading time* $U^{cts} : [0,T]\times[0,\infty)^2\times(0,\infty) \to [0,\infty)$ as

$$U^{cts}(t,\delta,x,\kappa) := \inf_{\Theta\in\mathcal{A}_t^{cts}(x)} J(t,\delta,\Theta,\kappa) \tag{6}$$

and the *value function for discrete trading time* as

$$U^{dis}(t,\delta,x,\kappa) := \inf_{\Theta\in\mathcal{A}_t^{dis}(x)} J(t,\delta,\Theta,\kappa) \geq U^{cts}(t,\delta,x,\kappa). \tag{7}$$

Denoting $\xi_n := \xi_{t_n} = \Delta\Theta_{t_n}$, we can also write the discrete time cost integral as a sum

$$U^{dis}(t,\delta,x,\kappa) = \inf_{\Theta\in\mathcal{A}_t^{dis}(x)} \mathbb{E}_{t,\delta,\kappa}\left[\sum_{t_n\geq t}\left(D_{t_n} + \frac{K_{t_n}}{2}\xi_n\right)\xi_n\right]. \tag{8}$$

Both value functions $U = U^{cts}$ and $U = U^{dis}$ fulfil the boundary conditions

$$U(T,\delta,x,\kappa) = \left(\delta + \frac{\kappa}{2}x\right)x \text{ and } U(t,\delta,0,\kappa) = 0. \tag{9}$$

Going forward we will use U and $\mathcal{A}_t(x)$ as a notation to indicate that the corresponding statement holds for both the continuous and discrete time case. If a certain statement is referring to only one setting, then we will explicitly use U^{cts} and $\mathcal{A}_t^{cts}(x)$ respectively U^{dis} and $\mathcal{A}_t^{dis}(x)$.

[1] Let us briefly recall how the right-hand side of (5) comes into play. Let the best ask price process (A_s) be modelled as $A_s = A_s^u + D_s$, where the *unaffected* best ask price (A_s^u) is a càdlàg $\mathcal{H}^1$-martingale. Then, given that the limit order book has the block form, the total cost of a strategy $\Theta \in \mathcal{A}_t^{cts}(x)$ is $\int_{[t,T]}\left(A_s + \frac{K_s}{2}\Delta\Theta_s\right) d\Theta_s$. A calculation involving integration by parts reveals that the expected total cost equals

$$\mathbb{E}_{t,\delta,\kappa}\left[A_T^u\Theta_{T+}\right] + \mathbb{E}_{t,\delta,\kappa}\left[\int_{[t,T]}\left(D_s + \frac{K_s}{2}\Delta\Theta_s\right) d\Theta_s\right] = A_t^u x + J(t,\delta,\Theta,\kappa)$$

with $J(t,\delta,\Theta,\kappa)$ given by (5) (notice that $\mathbb{E}_{t,\delta,\kappa}\int_{[t,T]}\Theta_s\, dA_s^u = 0$ because A^u is an $\mathcal{H}^1$-martingale and Θ is bounded). The first summand in the latter formula describes the expected cost that occurs due to trading in the unaffected price. This cost depends on the strategy $\Theta \in \mathcal{A}_t^{cts}(x)$ only through the total number of shares x that the strategy acquires, and, due to the martingale property of A^u, the expression is trivial: the initial price times the number of shares. The second summand in the latter formula describes the expected liquidity cost, which occurs due to price impact. This cost significantly depends on the strategy and is the object of our study.

The following simple result is recalled from Fruth, Schöneborn, and Urusov (2014). It shows that our formulas for the price impact and for the cost are economically sensible. This result will be essential below.

Lemma 2.1 (Splitting argument).
Doing two separate trades $\xi_\alpha, \xi_\beta > 0$ at the same time s has the same effect as trading at once $\xi := \xi_\alpha + \xi_\beta$, i.e. both alternatives incur the same cost and the same price deviation D_{s+}.

Proof. The cost is in both cases

$$\begin{aligned}\left(D_s + \frac{K_s}{2}\xi\right)\xi &= D_s(\xi_\alpha + \xi_\beta) + \frac{K_s}{2}(\xi_\alpha^2 + 2\xi_\alpha\xi_\beta + \xi_\beta^2) \\ &= \left(D_s + \frac{K_s}{2}\xi_\alpha\right)\xi_\alpha + \left(D_s + K_s\xi_\alpha + \frac{K_s}{2}\xi_\beta\right)\xi_\beta\end{aligned}$$

and the price deviation $D_{s+} = D_s + K_s(\xi_\alpha + \xi_\beta)$ after the trade is the same in both cases as well. □

Finally, let us relate the setting in this paper with that in Fruth, Schöneborn, and Urusov (2014). To this end, let the illiquidity coefficient be described by a deterministic strictly positive Borel function $k\colon [0,T] \to (0,\infty)$. We introduce the cost and the value functions

$$J_{k(\cdot)}(\Theta) \ (\equiv J_{k(\cdot)}(t,\delta,\Theta)), \quad U^{cts}_{k(\cdot)}(t,\delta,x), \quad U^{dis}_{k(\cdot)}(t,\delta,x)$$

similarly to (5)–(8) using the illiquidity k in place of K. These are the corresponding cost and value functions in Fruth, Schöneborn, and Urusov (2014) (notice that in this case, since k is deterministic, the infima over deterministic and adapted strategies coincide). Again, we will use just the notation $U_{k(\cdot)}$ to indicate that the corresponding statement holds for both the continuous and discrete time case. The following lemma is sometimes useful for performing comparisons with the case of deterministically changing illiquidity.

Lemma 2.2 (Stochastic versus deterministic illiquidity).
For all $t \in [0,T]$, $\delta \geq 0$, $x \geq 0$, $\kappa > 0$, we have

$$U(t,\delta,x,\kappa) \leq U_{\mathbb{E}_{t,\kappa}[K(\cdot)]}(t,\delta,x).$$

Proof. $U(t,\delta,x,\kappa)$ is smaller than or equal to the infimum like the one in (6) respectively (7), but over deterministic strategies. The latter infimum equals $U_{\mathbb{E}_{t,\kappa}[K(\cdot)]}(t,\delta,x)$ due to (3) and (5). □

3 Definition of WR-BR structure

In this section we define the *WR-BR* structure (WR: wait region, BR: buy region) and derive fundamental properties. A detailed introduction of the WR-BR structure is provided by Fruth, Schöneborn, and Urusov (2014); we therefore keep the exposition brief in this section. In particular, we do not prove Proposition 3.2 below, since the proof is similar to the corresponding proof in the aforementioned paper.

Before attacking the formal definition of WR-BR structure, we note that the four-dimensional value function U can be reduced by one dimension due to the following scaling property (its proof is straightforward).

Lemma 3.1 (Optimal strategies scale linearly).
For all $a \in [0,\infty)$ we have

$$U(t,a\delta,ax,\kappa) = a^2 U(t,\delta,x,\kappa). \tag{10}$$

Furthermore, if $\Theta^* \in \mathcal{A}_t(x)$ is optimal for $U(t,\delta,x,\kappa)$, then $a\Theta^* \in \mathcal{A}_t(ax)$ is optimal for $U(t,a\delta,ax,\kappa)$.

We will also need two useful results:

Proposition 3.2 (Continuity of the value function)**.**
For each $t \in [0,T]$ *and* $\kappa > 0$*, the function*

$$U(t,\cdot,\cdot,\kappa)\colon [0,\infty)^2 \to [0,\infty)$$

is continuous.

Proposition 3.3 (Trading never completes early)**.**
For all $t \in [0,T)$*,* $\delta \geq 0$*,* $x > 0$ *and* $\kappa > 0$*, the value function satisfies*

$$U(t,\delta,x,\kappa) < \left(\delta + \frac{\kappa}{2}x\right)x,$$

i.e. it is never optimal to buy the whole remaining position at any time $t \in [0,T)$*.*

Proof. The result immediately follows from Lemma 2.2 and the corresponding result for deterministically varying K, see Proposition 5.6 in Fruth, Schöneborn, and Urusov (2014). □

For $\delta > 0$, we can take $a = \frac{1}{\delta}$ and apply Lemma 3.1 to get

$$\begin{aligned} U(t,\delta,x,\kappa) &= \delta^2 U\left(t,1,\frac{x}{\delta},\kappa\right) = \delta^2 V(t,y,\kappa) \quad \text{with} \\ y &:= \frac{x}{\delta}, \\ V(t,y,\kappa) &:= U(t,1,y,\kappa), \quad V(T,y,\kappa) = y + \frac{\kappa}{2}y^2, \quad V(t,0,\kappa) \equiv 0. \end{aligned} \tag{11}$$

Going forward we will use V^{cts} and V^{dis} where we need to differentiate between continuous and discrete time settings. We now see that the function $U(t,\delta_{fix},x,\kappa)$ for some $\delta_{fix} > 0$ or $U(t,\delta,x_{fix},\kappa)$ for some $x_{fix} > 0$ already determines the entire value function. In the following we will often analyze the function V in order to derive the properties of U. Technically this does not directly allow us to draw conclusions for $U(t,0,x,\kappa)$, since, for $\delta = 0$, the ratio $y = x/\delta$ is not defined. The extension of our proofs to allow the possibility $\delta = 0$ is however straightforward by a continuity argument (see Proposition 3.2).

We first define the buy and wait region and subsequently define the barrier function.

Definition 3.4 (Buy and wait region)**.**
For any $t \in [0,T]$ and $\kappa > 0$, we define the *inner buy region* as

$$Br_{t,\kappa} := \left\{ y \in (0,\infty) \,|\, \exists \xi \in (0,y)\colon U(t,1,y,\kappa) = U\left(t, 1+\kappa\xi, y-\xi, \kappa\right) + \left(1 + \frac{\kappa}{2}\xi\right)\xi \right\},$$

and call the following sets the *buy region* and *wait region* at time t for the illiquidity coefficient κ:

$$BR_{t,\kappa} := \overline{Br_{t,\kappa}}, \quad WR_{t,\kappa} := [0,\infty) \setminus Br_{t,\kappa}$$

(the bar indicates closure in $\mathbb{R}$).

The inner buy region at time t for illiquidity coefficient κ hence consists of all values y such that immediate buying at the state $(1,y)$ is value preserving. The wait region on the other hand contains all values y such that any non-zero purchase at $(1,y)$ destroys value. Let us note that $Br_{T,\kappa} = (0,\infty)$, $BR_{T,\kappa} = [0,\infty)$ and $WR_{T,\kappa} = \{0\}$. The wait region / buy region conjecture can now be formalized as follows.

Definition 3.5 (WR-BR structure).
The value function U has *WR-BR structure* if there exists a *barrier function*

$$c\colon [0,T] \times (0,\infty) \to [0,\infty]$$

such that for all $t \in [0,T]$ and $\kappa > 0$,

$$Br_{t,\kappa} = (c(t,\kappa), \infty)$$

with the convention $(\infty,\infty) := \emptyset$. For the value function U^{dis} in discrete time to have WR-BR structure, we only consider $t \in \{t_0, ..., t_N\}$ and set $c^{dis}(t,\kappa) = \infty$ for $t \notin \{t_0, ..., t_N\}$.

It is worth noting that the barrier can be infinite even in continuous time or in discrete time at time points $t_0, \ldots, t_{N-1}$, that is, there can be certain t and κ, for which it is never optimal to perform a block trade, regardless of how large the remaining position is. We refer to Propositions 5.8 and 5.9 in Fruth, Schöneborn, and Urusov (2014) for sufficient conditions for infinite barrier in the case of deterministically varying K.

Let us remark that we always have $c(T,\kappa) = 0$. On the contrary, the barrier is always strictly positive for $t \in [0,T)$ (whenever the value function U has WR-BR structure):

Proposition 3.6 (Wait region near zero).
Assume that the value function U has WR-BR structure with the barrier c. Then, for any $t \in [0,T)$ and $\kappa > 0$, we have $c(t,\kappa) \in (0,\infty]$.

Proof. Assume that for some $t \in [0,T)$ and $\kappa > 0$ we have $c(t,\kappa) = 0$. Let us fix some $y > 0$ and define

$$\bar{\xi} := \sup\left\{\xi \in (0,y) \mid U(t,1,y,\kappa) = U(t, 1+\kappa\xi, y-\xi, \kappa) + \left(1 + \frac{\kappa}{2}\xi\right)\xi\right\} \leq y.$$

Since $U(t,\cdot,\cdot,\kappa)$ is continuous (Proposition 3.2), we get

$$U(t,1,y,\kappa) = U(t, 1+\kappa\bar{\xi}, y-\bar{\xi}, \kappa) + \left(1 + \frac{\kappa}{2}\bar{\xi}\right)\bar{\xi}. \tag{12}$$

If $\bar{\xi} < y$, then, due to the scaling property of Lemma 3.1, the fact that $(y-\bar{\xi})/(1+\kappa\bar{\xi}) \in Br_{t,\kappa}$, and the splitting argument of Lemma 2.1, we arrive at a contradiction with the definition of $\bar{\xi}$. Thus, $\bar{\xi} = y$, but then formula (12) contradicts Proposition 3.3. This completes the proof. ☐

The following proposition characterizes the WR-BR structure and will be needed for some of our main results.

Proposition 3.7. (WR-BR structure is equivalent to trading towards the barrier).
Assume that for each (t,δ,x,κ) there exists a unique optimal strategy

$$(\Theta^*_s(t,\delta,x,\kappa))_{s\in[t,T]} \in \mathcal{A}_t(x).$$

Then the following statements are equivalent.

(a) The value function has WR-BR structure.

(b) There exists $c : [0,T) \times (0,\infty) \to (0,\infty]$ such that for all (t,δ,x,κ)

$$\Delta\Theta^*_t(t,\delta,x,\kappa) = \max\left\{0, \frac{x - c(t,\kappa)\delta}{1+\kappa c(t,\kappa)}\right\}. \tag{13}$$

In particular, $\Delta\Theta^*_t(t,\delta,x,\kappa)$ is continuous in δ and x.

(c) For all (t,δ,κ), the function $x \mapsto \Delta\Theta^*_t(t,\delta,x,\kappa)$ is increasing.

Proof. First we prove the equivalence of (a) and (b). Statement (c) follows immediately from (b). We conclude by showing that (c) implies (b). The scaling property (Lemma 3.1) yields

$$\Delta\Theta_t^*(t,\delta,x,\kappa) = \delta\Delta\Theta_t^*\left(t,1,\frac{x}{\delta},\kappa\right).$$

Therefore we only need to discuss the case $\delta = 1$. Fix arbitrary $t \in [0,T]$, $\kappa \in (0,\infty)$.

(a) $\Rightarrow$ (b) The assertion holds for $x = 0$. Assume $x \in (0, c(t,\kappa)]$. Then the WR-BR structure implies that for all $\xi \in (0,x)$

$$U(t,1,x,\kappa) < U\left(t,1+\kappa\xi,x-\xi,\kappa\right) + \left(1+\frac{\kappa}{2}\xi\right)\xi.$$

Therefore it cannot be optimal to trade immediately at time t.

Assume $c(t,\kappa) < \infty$ and $x \in (c(t,\kappa),\infty)$. Then the WR-BR structure implies that there exists $\tilde{\xi} \in (0,x)$ such that

$$U(t,1,x,\kappa) = U\left(t,1+\kappa\tilde{\xi},x-\tilde{\xi},\kappa\right) + \left(1+\frac{\kappa}{2}\tilde{\xi}\right)\tilde{\xi}.$$

Due to the uniqueness of the optimal strategy, we get

$$\Delta\Theta_t^*(t,1,x,\kappa) = \tilde{\xi} + \Delta\Theta_t^*\left(t,1+\kappa\tilde{\xi},x-\tilde{\xi},\kappa\right) > 0.$$

For $\tilde{\xi} < \frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)}$, we have $\frac{x-\tilde{\xi}}{1+\kappa\tilde{\xi}} > c(t,\kappa)$ and thus

$$\Delta\Theta_t^*\left(t,1+\kappa\tilde{\xi},x-\tilde{\xi},\kappa\right) > 0.$$

Consequently, $\Delta\Theta_t^*(t,1,x,\kappa) \geq \frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)}$. Two trades executed immediately after each other have the same effect as one trade of their combined size (see Lemma 2.1). Due to this splitting argument, we have

$$\Delta\Theta_t^*(t,1,x,\kappa) = \frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)} + \Delta\Theta_t^*\left(t,1+\kappa\frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)},x-\frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)},\kappa\right).$$

Observe that the second summand equals zero because

$$\frac{x-\frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)}}{1+\kappa\frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)}} = c(t,\kappa).$$

(b) $\Rightarrow$ (a) Assume $x \in (0, c(t,\kappa)]$. Then (13) implies $\Delta\Theta_t^*(t,1,x,\kappa) = 0$. Together with the uniqueness of the optimal strategy we can therefore conclude that $x \notin Br_{t,\kappa}$, since for all $\xi \in (0,x)$

$$U(t,1,x,\kappa) < U\left(t,1+\kappa\xi,x-\xi,\kappa\right) + \left(1+\frac{\kappa}{2}\xi\right)\xi.$$

Assume $c(t,\kappa) < \infty$ and $x \in (c(t,\kappa),\infty)$. Then (13) implies

$$\Delta\Theta_t^*(t,1,x,\kappa) \in (0,x).$$

The optimality of Θ^* leads to the conclusion $x \in Br_{t,\kappa}$ since

$$\begin{aligned} U(t,1,x,\kappa) &= U\left(t,1+\kappa\Delta\Theta_t^*(t,1,x,\kappa),x-\Delta\Theta_t^*(t,1,x,\kappa),\kappa\right) \\ &\quad + \left(1+\frac{\kappa}{2}\Delta\Theta_t^*(t,1,x,\kappa)\right)\Delta\Theta_t^*(t,1,x,\kappa). \end{aligned}$$

(c) $\Rightarrow$ (b) Define

$$c(t,\kappa) := \inf\{x \in (0,\infty) | \Delta\Theta_t^*(t,1,x,\kappa) > 0\}.$$

We are done for $c(t,\kappa) = \infty$. Let $c(t,\kappa) < \infty$. Then the definition of $c(t,\kappa)$ guarantees $\Delta\Theta_t^*(t,1,x,\kappa) = 0$ for all $x < c(t,\kappa)$, and Property (c) implies $\Delta\Theta_t^*(t,1,x,\kappa) > 0$ for all $x > c(t,\kappa)$. Suppose for a contradiction that

$$\Delta\Theta_t^*(t,1,c(t,\kappa),\kappa) > 0.$$

Due to the uniqueness and the splitting argument, we then have, for $\epsilon \in (0, \Delta\Theta_t^*(t,1,c(t,\kappa),\kappa))$,

$$\Delta\Theta_t^*(t,1,c(t,\kappa),\kappa) = \epsilon + \Delta\Theta_t^*(t,1+\kappa\epsilon, c(t,\kappa)-\epsilon,\kappa) = \epsilon < \Delta\Theta_t^*(t,1,c(t,\kappa),\kappa).$$

Therefore, $\Delta\Theta_t^*(t,1,x,\kappa) = 0$ for all $x \le c(t,\kappa)$.

We still need to prove $\Delta\Theta_t^*(t,1,x,\kappa) = \frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)}$ for $x > c(t,\kappa)$. Let us first assume that $\Delta\Theta_t^*(t,1,x,\kappa) > \frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)}$. Once more, we make use of the uniqueness and the splitting argument in order to get a contradiction

$$\begin{aligned}\Delta\Theta_t^*(t,1,x,\kappa) &= \frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)} + \Delta\Theta_t^*\left(t, 1+\kappa\frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)}, x - \frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)}, \kappa\right) \\ &= \frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)} < \Delta\Theta_t^*(t,1,x,\kappa).\end{aligned}$$

Finally, assume $\Delta\Theta_t^*(t,1,x,\kappa) < \frac{x-c(t,\kappa)}{1+\kappa c(t,\kappa)}$. That is, $\frac{x-\Delta\Theta_t^*(t,1,x,\kappa)}{1+\kappa\Delta\Theta_t^*(t,1,x,\kappa)} > c(t,\kappa)$ and we again arrive at a contradiction:

$$\begin{aligned}\Delta\Theta_t^*(t,1,x,\kappa) &= \Delta\Theta_t^*(t,1,x,\kappa) + \Delta\Theta_t^*\Big(t, 1+\kappa\Delta\Theta_t^*(t,1,x,\kappa), x - \Delta\Theta_t^*(t,1,x,\kappa), \kappa\Big) \\ &> \Delta\Theta_t^*(t,1,x,\kappa).\end{aligned}$$

□

4 The WR-BR theorem

In this section we show that the value function exhibits WR-BR structure if K is a diffusion satisfying the following assumption.

Assumption 4.1. (Special diffusion).
K is a (possibly time-inhomogeneous) diffusion

$$dK_s = \mu(s,K_s)\,ds + \sigma(s,K_s)\,dW_s^K, \quad K_t = \kappa > 0, \tag{14}$$

for an $(\mathcal{F}_s)$-Brownian motion W^K and $\mu,\sigma\colon [0,T]\times(0,\infty)\to\mathbb{R}$ such that, for any initial time $t \in [0,T]$ and starting point $K_t = \kappa > 0$, the stochastic differential equation has a weak solution which is unique in law, is strictly positive and has finite first moments. Furthermore, for all $t \in [0,T]$ and $\kappa > 0$, we have

i) $\eta_s := \frac{2\rho_s}{K_s} + \frac{\mu(s,K_s)}{K_s^2} - \frac{\sigma^2(s,K_s)}{K_s^3} > 0$ $\mathbb{P}_{t,\kappa}\times\mu_L$-a.e. on $\Omega\times[t,T]$ (μ_L denotes the Lebesgue measure),

ii) $\mathbb{E}_{t,\kappa}\left[\frac{\sup_{s\in[t,T]} K_s^2}{\inf_{s\in[t,T]} K_s}\right] < \infty$,

iii) $\mathbb{E}_{t,\kappa}\left[\left(\int_0^T |\eta_s|\,ds\right)\left(\sup_{s\in[t,T]} K_s^2\right)\right] < \infty$.

In Section 7 below we study profitable round trip strategies without assuming that the process η is positive, but we will need part iii) of Assumption 4.1. That is why we write absolute value of η in iii).

Theorem 4.2. (WR-BR theorem).
If Assumption 4.1 holds, then there is a unique optimal strategy, and we have WR-BR structure.

In fact, we will see in the proof that existence and uniqueness of the optimal strategy both in discrete and continuous time as well as WR-BR structure in discrete time hold under parts i)–ii) of Assumption 4.1. We need part iii) only for WR-BR structure in continuous time.

We prove Theorem 4.2 in two steps. In Subsection 4.1, we show that Assumption 4.1 ensures strict convexity of the cost functional J in the strategy, which in turn guarantees existence and uniqueness of the optimal strategy. As we show in Subsection 4.2, the uniqueness excludes WR-BR-WR and other situations: at any upper boundary of a buy region it must be equally optimal to wait as it is to execute the strictly positive trade to the lower boundary of the buy region. We first pursue this line of argument for the discrete time case and then transfer it to continuous time and thus do not use the Hamilton-Jacobi-Bellman equation.

Part i) of Assumption 4.1 is the most critical in the proof since it is directly linked to the convexity of J. As we will see in Section 7, it is also related to the absence of profitable round trip trading strategies in a two-sided order book model. Parts ii) and iii) of Assumption 4.1 are required for more technical aspects of our proof.

Theorem 4.2 does not cover all models which result in a WR-BR structure.[2] In Section 6 we provide examples violating the WR-BR structure, highlighting that some assumptions on K are necessary to guarantee a WR-BR structure.

4.1 Existence of a unique optimal strategy

Under Assumption 4.1, we show in Lemma 4.3 that $J(\Theta)$ is strictly convex. This guarantees the uniqueness of an optimal strategy provided it exists. We can then use the convexity together with the Komlós theorem to finally get the existence of an optimal strategy in Proposition 4.4.

Lemma 4.3. (Costs are convex in the strategy).
Let Assumption 4.1 hold. Then, for all $t \in [0,T]$, $\delta \geq 0$ and $\kappa > 0$, the function $J(\cdot) \equiv J(t,\delta,\cdot,\kappa)$ is finite and strictly convex on $\mathcal{A}_t$.

Proof. Let t, δ and κ be fixed. Clearly, Assumption 4.1 ii) implies $\mathbb{E}_{t,\kappa} \sup_{s\in[t,T]} K_s < \infty$, hence $J(\cdot)$ is finite on the whole $\mathcal{A}_t$. We demonstrate below that

$$J(\Theta) = \frac{1}{2}\mathbb{E}_{t,\delta,\kappa}\left[\frac{D_{T+}^2}{K_T} - \frac{\delta^2}{\kappa} + \int_{[t,T]} \eta_s D_s^2 ds\right] \tag{15}$$

with (η_s) as in Assumption 4.1 i). The right-hand side is strictly convex in the process $(D_s)_{s\in[t,T]}$. Thus, for two different strategies $\Theta', \Theta'' \in \mathcal{A}_t$ with corresponding D', D'' both starting in $D'_t = D''_t = \delta$, we have $D(\nu\Theta' + (1-\nu)\Theta'') = \nu D' + (1-\nu)D''$, hence $J(\nu\Theta' + (1-\nu)\Theta'') < \nu J(\Theta') + (1-\nu)J(\Theta'')$ for all $\nu \in (0,1)$ as desired. Hence, we only need to show (15).

Define the local martingale $M_s := \int_{[t,s\wedge T]} \frac{D_u^2\sigma(u,K_u)}{2K_u^2} dW_u^K$ for $s \in [t,\infty)$. That is, $\tau_n = \{s \geq t \mid \langle M\rangle_s \geq n\}$ is an increasing sequence of stopping times such that $\tau_n \nearrow \infty$ a.s. and M^{τ_n} is a martingale for

[2] Restricting trading to only two points in time is an example which always has WR-BR structure irrespective of Assumption 4.1. Furthermore, in the case of deterministically varying K we always have WR-BR structure in discrete time and, for continuous K, in continuous time, see Fruth, Schöneborn, and Urusov (2014).

every n. In particular, $\mathbb{E}_{t,\delta,\kappa}[M_{T\wedge\tau_n}] = 0$. Due to the monotone convergence theorem and $\tau_n \geq T$ a.s. for large n,

$$J(\Theta) = \lim_{n\to\infty} \mathbb{E}_{t,\delta,\kappa}\left[\int_{[t,T\wedge\tau_n]} \left(D_s + \frac{K_s}{2}\Delta\Theta_s\right) d\Theta_s\right]. \tag{16}$$

Using $d\Theta_s = \frac{dD_s + \rho_s D_s ds}{K_s}$ and $\Delta\Theta_s = \frac{\Delta D_s}{K_s}$, we get

$$J(\Theta) = \lim_{n\to\infty} \mathbb{E}_{t,\delta,\kappa}\left[\int_{[t,T\wedge\tau_n]} \frac{D_s + \frac{1}{2}\Delta D_s}{K_s} dD_s + \int_{[t,T\wedge\tau_n]} \frac{\rho_s D_s^2}{K_s} ds + \int_{[t,T\wedge\tau_n]} \frac{\frac{1}{2}\Delta D_s \rho_s D_s}{K_s} ds\right].$$

The last integral is zero, since D has at most countably many jumps. With integration by parts for càglàd processes,

$$\int_{[t,T\wedge\tau_n]} \frac{D_s}{K_s} dD_s = \frac{D^2_{(T\wedge\tau_n)+}}{K_{(T\wedge\tau_n)}} - \frac{\delta^2}{\kappa} - \int_{[t,T\wedge\tau_n]} D_s d\left(\frac{D}{K}\right)_s - \sum_{s\in[t,T\wedge\tau_n]} \frac{(\Delta D_s)^2}{K_s}.$$

Use $d\left(\frac{D}{K}\right)_s = \frac{1}{K_s} dD_s + D_s d\left(\frac{1}{K_s}\right)$ and rearrange terms to get

$$\int_{[t,T\wedge\tau_n]} \frac{D_s}{K_s} dD_s = \frac{1}{2}\left(\frac{D^2_{(T\wedge\tau_n)+}}{K_{(T\wedge\tau_n)}} - \frac{\delta^2}{\kappa} - \int_{[t,T\wedge\tau_n]} D_s^2 d\left(\frac{1}{K_s}\right) - \sum_{s\in[t,T\wedge\tau_n]} \frac{(\Delta D_s)^2}{K_s}\right).$$

Applying Itô's formula

$$d\left(\frac{1}{K_s}\right) = \left(\frac{\sigma^2(s,K_s)}{K_s^3} - \frac{\mu(s,K_s)}{K_s^2}\right) ds - \frac{\sigma(s,K_s)}{K_s^2} dW_s^K$$

yields

$$\int_{[t,T\wedge\tau_n]} \left(D_s + \frac{K_s}{2}\Delta\Theta_s\right) d\Theta_s = \frac{1}{2}\left[\frac{D^2_{(T\wedge\tau_n)+}}{K_{T\wedge\tau_n}} - \frac{\delta^2}{\kappa} + \int_{[t,T\wedge\tau_n]} \eta_s D_s^2 ds + M_{T\wedge\tau_n}\right].$$

The assertion follows, since Lebesgue's dominated convergence theorem together with Assumption 4.1 ii) guarantee

$$\mathbb{E}_{t,\delta,\kappa}\left[\frac{D^2_{(T\wedge\tau_n)+}}{K_{T\wedge\tau_n}}\right] \xrightarrow[n\to\infty]{} \mathbb{E}_{t,\delta,\kappa}\left[\frac{D^2_{T+}}{K_T}\right], \tag{17}$$

while, by the monotone convergence theorem, we have

$$\mathbb{E}_{t,\delta,\kappa}\left[\int_{[t,T\wedge\tau_n]} \eta_s D_s^2 ds\right] \xrightarrow[n\to\infty]{} \mathbb{E}_{t,\delta,\kappa}\left[\int_{[t,T]} \eta_s D_s^2 ds\right]. \tag{18}$$

□

Proposition 4.4. (Existence and uniqueness of an optimal strategy).
Let Assumption 4.1 hold. Then, for all $t \in [0,T]$, $\delta \geq 0$, $x \geq 0$ and $\kappa > 0$, there exists a unique optimal strategy, i.e. there exists a unique $\Theta^* = \Theta^*(t,\delta,x,\kappa) \in \mathcal{A}_t(x)$ with

$$J(t,\delta,\Theta^*,\kappa) = \inf_{\Theta\in\mathcal{A}_t(x)} J(t,\delta,\Theta,\kappa).$$

Proof. Thanks to Lemma 4.3, we only need to prove existence. Let t, δ and κ be fixed. We start by showing that there exists a sequence of strategies $\left(\overline{\Theta}^n\right) \subset \mathcal{A}_t(x)$ that converges in some sense to a strategy $\Theta^* \in \mathcal{A}_t(x)$ and minimizes the cost J, i.e. $\lim_{n\to\infty} J\left(\overline{\Theta}^n\right) = \inf_{\Theta\in\mathcal{A}_t(x)} J(\Theta)$. We conclude by deducing that $\lim_{n\to\infty} J(\overline{\Theta}^n) = J(\Theta^*)$.

Let $\left(\Theta^j\right) \subset \mathcal{A}_t(x)$ be a minimizing sequence for J. Due to the Komlós theorem in the form of Lemma 3.5 from Kabanov (1999), there exists a Cesaro convergent subsequence (Θ^{j_m}). That is,

$$\overline{\Theta}^n := \frac{1}{n}\sum_{m=1}^{n} \Theta^{j_m}$$

converges to some strategy $\Theta^* \in \mathcal{A}_t$ in the following sense. For $\mathbb{P}_{t,\kappa}$-almost every ω, the measures $\overline{\Theta}^n(\omega)$ on $[t,T]$ converge weakly to the measure $\Theta^*(\omega)$. In what follows we call such a convergence *pathwise weak convergence in time.* Equivalently, for almost every ω, we have $\lim_{n\to\infty} \overline{\Theta}^n_s = \Theta^*_s$ whenever $s \in [t,T]$ with $\Delta\Theta^*_s = 0$. We set $\Theta^*_{T+} = x$ redefining Θ^*_{T+} if necessary. Notice that this does not disturb the weak convergence. Thus, $\Theta^* \in \mathcal{A}_t(x)$. Moreover, $\left(\overline{\Theta}^n\right) \subset \mathcal{A}_t(x)$ is again a minimizing sequence for J, since J is convex.

It remains to show that Θ^* attains the infimum. Applying (15) yields

$$J\left(\overline{\Theta}^n\right) = \frac{1}{2}\mathbb{E}_{t,\delta,\kappa}\left[\frac{\left(D^n_{T+}\right)^2}{K_T} - \frac{\delta^2}{\kappa} + \int_{[t,T]} \eta_s \left(D^n_s\right)^2 ds\right], \tag{19}$$

$$J\left(\Theta^*\right) = \frac{1}{2}\mathbb{E}_{t,\delta,\kappa}\left[\frac{\left(D^*_{T+}\right)^2}{K_T} - \frac{\delta^2}{\kappa} + \int_{[t,T]} \eta_s \left(D^*_s\right)^2 ds\right], \tag{20}$$

where D^n and D^* are the price impact processes that correspond to $\overline{\Theta}^n$ and Θ^*. By the (pathwise weak in time) convergence of $\overline{\Theta}^n$ to Θ^*, for almost every ω, we get $\lim_{n\to\infty} D^n_s = D^*_s$ for every point $s \in [t,T]$, where Θ^* is continuous, as well as for $s = T+$.[3] Fatou's lemma and (19)–(20) now imply $J\left(\Theta^*\right) \le \liminf_{n\to\infty} J\left(\overline{\Theta}^n\right)$, which means that Θ^* is an optimal strategy. □

4.2 Wait and buy region structure

Under Assumption 4.1, we will now exploit the uniqueness of the optimal strategy to prove WR-BR structure. Proposition 4.5 treats the discrete time case, which is then transferred to continuous time in Proposition 4.8.

Proposition 4.5. (Discrete time: WR-BR structure).
Let Assumption 4.1 hold. Then the value function U^{dis} has WR-BR structure.

Proof. According to Propositions 3.7 and 4.4, we only need to show that the optimal initial trade $\Delta\Theta^*_{t_n}(t_n,\delta,x,\kappa)$ is increasing in x, where Θ^* denotes the corresponding optimal strategy. Due to the scaling property of the value function (Lemma 3.1),

$$\Delta\Theta^*_{t_n}(t_n,\delta,x,\kappa) = \delta\Delta\Theta^*_{t_n}\left(t_n,1,\frac{x}{\delta},\kappa\right).$$

Due to the splitting argument (Lemma 2.1) and the uniqueness of the optimal strategy, $\Delta\Theta^*_{t_n}(t_n,1,\cdot,\kappa)$ must be increasing and continuous apart from a possible discontinuity in the form of a jump back to zero. That is there might exist $y>0$ with $\Delta\Theta^*_{t_n}(t_n,1,y-,\kappa) > 0$ and $\Delta\Theta^*_{t_n}(t_n,1,y+,\kappa) = 0$. In the following, we exclude such discontinuities using a Komlós argument as in the proof of Proposition 4.4.

Suppose for a contradiction that such a discontinuity exists in $y>0$. Let us take some monotone sequences $y^{1,j} \nearrow y$ and $y^{2,j} \searrow y$ and define $\Theta^{i,j} := \Theta^*(t_n,1,y^{i,j},\kappa)$ for $i \in \{1,2\}$. Let us choose $\epsilon > 0$

[3]See also Lemma 7.3 of Fruth, Schöneborn, and Urusov (2014).

such that $\Delta\Theta^{1,j}_{t_n} \geq \epsilon > 0$ for all sufficiently large j. Without loss of generality we assume that the latter inequality holds for all j. Since V^{dis} is continuous in y (see Proposition 3.2),

$$J\left(t_n, 1, \Theta^{1,j}, \kappa\right) = V^{dis}\left(t_n, y^{1,j}, \kappa\right) \xrightarrow[j\to\infty]{} V^{dis}\left(t_n, y, \kappa\right).$$

Define $b_j := \frac{y}{y^{1,j}} \searrow 1$. Then we have

$$\begin{aligned} 0 &\leq J\left(t_n, 1, b_j\Theta^{1,j}, \kappa\right) - J\left(t_n, 1, \Theta^{1,j}, \kappa\right) \\ &\leq J\left(t_n, b_j, b_j\Theta^{1,j}, \kappa\right) - J\left(t_n, 1, \Theta^{1,j}, \kappa\right) = (b_j^2 - 1)J\left(t_n, 1, \Theta^{1,j}, \kappa\right) \xrightarrow[j\to\infty]{} 0. \end{aligned}$$

Therefore, $\left(b_j\Theta^{1,j}\right)$ is a minimizing sequence of strategies that build up the position of y shares, i.e., $b_j\Theta^{1,j} \in \mathcal{A}^{dis}_{t_n}(y)$ and

$$\lim_{j\to\infty} J\left(t_n, 1, b_j\Theta^{1,j}, \kappa\right) = V^{dis}\left(t_n, y, \kappa\right).$$

As in the proof of Proposition 4.4, we can define $\overline{\Theta} \in \mathcal{A}^{dis}_{t_n}(y)$ as the pathwise weak in time limit of the averaged sum over a subsequence of $\left(b_j\Theta^{1,j}\right)$ such that $J(t_n, 1, \overline{\Theta}, \kappa) = V^{dis}(t_n, y, \kappa)$, i.e. $\overline{\Theta}$ is an optimal strategy. Due to the construction of $\overline{\Theta}$, with $\epsilon > 0$ from above, we have

$$\Delta\overline{\Theta}_{t_n}\left(t_n, 1, y, \kappa\right) \geq \epsilon > 0.$$

Similarly, one constructs an optimal strategy $\overline{\overline{\Theta}} \in \mathcal{A}^{dis}_{t_n}(y)$ using the sequence $\left(\frac{y}{y^{2,j}}\Theta^{2,j}\right)$ of strategies with zero initial trade. Since we now treat the discrete time case, the initial trade remains zero also in the weak limit:

$$\Delta\overline{\overline{\Theta}}_{t_n}\left(t_n, 1, y, \kappa\right) = 0.$$

Thus, $\overline{\Theta}$ and $\overline{\overline{\Theta}}$ are different. This contradicts the uniqueness of the optimal strategy. $\square$

The line of argument used in the preceding proof does not extend directly to continuous time. Let us also notice that we did not yet use part iii) of Assumption 4.1. We now transfer the discrete time result of Proposition 4.5 to continuous time in Proposition 4.8 using the approximation techniques of Lemmas 4.6 and 4.7, and we will now need part iii) of Assumption 4.1.

Lemma 4.6. (Approximation via step functions).
Let Assumption 4.1 hold. For $\Theta \in \mathcal{A}^{cts}_t(x)$, let $\Theta^N \in \mathcal{A}^{dis}_t(x)$ be its approximation from below by an equidistant grid step function. More precisely, define $\mathcal{T}^0_t := \{t, T\}$,

$$\mathcal{T}^{N+1}_t := \mathcal{T}^N_t \cup \left\{ \left(s + \frac{T-t}{2^{N+1}}\right) \wedge T \mid s \in \mathcal{T}^N_t \right\}$$

and

$$\Theta^N_s := \left\{ \begin{array}{lll} 0 & \text{if } s = t \\ \Theta_{u+} & \text{if } s \in \left(u, u + \frac{T-t}{2^N}\right], u \in \mathcal{T}^N_t \\ x & \text{if } s = T+ \end{array} \right\}.$$

Then $J(t, 1, \Theta, \kappa) = \lim_{N\to\infty} J\left(t, 1, \Theta^N, \kappa\right)$.

Proof. We proceed as at the end of the proof of Proposition 4.4. That is we only need to show that Θ^N converges pathwise weakly in time to Θ. Due to $\mathcal{T}^N_t \subset \mathcal{T}^{N+1}_t$, Θ^N is increasing in N. For all $s \in [t, T+]$, the sequence $\left(\Theta^N_s\right)_{N\in\mathbb{N}}$ is bounded above by Θ_s. Hence, it is convergent. Due to the definition of Θ^N, we must even have $\lim_{N\to\infty} \Theta^N_s = \Theta_s$ for all $s \in [t, T]$ with $\Delta\Theta_s = 0$. Now the result follows from (15) and the dominated convergence theorem (apply Assumption 4.1 ii) and iii)). $\square$

Lemma 4.7. (Cesaro weak convergence).
Fix $t \in [0,T], \kappa \in (0,\infty)$ and for various $x \in [0,\infty)$ consider

$$\left(\Theta^N(t,1,x,\kappa)\right)_{N\in\mathbb{N}} \subset \mathcal{A}_t^{cts}(x).$$

Then there exists a subsequence $N_j(t,\kappa)$, which does not depend on x, and a set of strategies $\tilde{\Theta}(t,1,\cdot,\kappa)$ such that for all $x \in [0,\infty)\cap\mathbb{Q}$

$$\frac{1}{m}\sum_{j=1}^{m}\Theta^{N_j}(t,1,x,\kappa) \xrightarrow[m\to\infty]{w} \tilde{\Theta}(t,1,x,\kappa). \tag{21}$$

In (21) the notation "$\xrightarrow{w}$" stands for the pathwise weak convergence in time (cf. the proof of Proposition 4.4).

Proof. Since $\mathbb{Q}$ is countable, we can write $[0,\infty)\cap\mathbb{Q} = \{x_1, x_2, ...\}$. For each $x \in [0,\infty)$, the Komlós theorem guarantees the existence of a subsequence $N_j(t,x,\kappa)$ such that the desired pathwise weak convergence in time holds. That is we get $(N_j^{(1)})_{j\in\mathbb{N}} \subset \mathbb{N}$ for x_1 and extract the subsequence $N_j^{(2)}$ for x_2 from $N_j^{(1)}$, etc. We remark that the Komlós theorem gives not only Cesaro convergent subsequences, but subsequences such that all their subsequences are Cesaro convergent to the same limit. The Cantor diagonal sequence $N_j := N_j^{(j)}$ then guarantees the Cesaro weak convergence of $\Theta^{N_j}(t,1,x,\kappa)$ for all $x \in [0,\infty)\cap\mathbb{Q}$. □

Proposition 4.8. (Continuous time: WR-BR structure).
Let Assumption 4.1 hold. Then the value function U^{cts} has WR-BR structure.

Proof. As in the proof of Proposition 4.5, we only need to exclude the jump back to zero of $x \mapsto \Delta\Theta_t^*(t,1,x,\kappa)$. Let $\Theta^N \in \mathcal{A}_t^{dis}(x)$ be the approximation of $\Theta^* \in \mathcal{A}_t^{cts}(x)$ by step functions from below as in Lemma 4.6. Then

$$J(t,1,\Theta^*,\kappa) = \lim_{N\to\infty} J\left(t,1,\Theta^N,\kappa\right).$$

Let Θ^{*N} be the unique optimal strategy within $\mathcal{A}_t^{dis}(x)$ for the time grid $\mathcal{T}_t^N$, i.e.

$$J\left(t,1,\Theta^N,\kappa\right) \geq J\left(t,1,\Theta^{*N},\kappa\right) \geq J\left(t,1,\Theta^*,\kappa\right).$$

Hence,

$$J(t,1,\Theta^*,\kappa) = \lim_{N\to\infty} J\left(t,1,\Theta^{*N},\kappa\right).$$

That is, for each $x \in [0,\infty)$, $(\Theta^{*N}(t,1,x,\kappa))_{N\in\mathbb{N}}$ is a minimizing sequence, and for each $N \in \mathbb{N}$, $x \mapsto \Delta\Theta_t^{*N}(t,1,x,\kappa)$ is increasing thanks to Proposition 4.5.

Apply Lemma 4.7 to $\Theta^{*N}(t,1,x,\kappa)$ (for all rational x). The resulting strategy $\tilde{\Theta}(t,1,x,\kappa)$ as in (21) is optimal (apply convexity of the cost function together with (15) and the dominated convergence theorem). Since the optimal strategy is unique, $\tilde{\Theta}(t,1,x,\kappa)$ must coincide with $\Theta^*(t,1,x,\kappa)$ for all $x \in [0,\infty)\cap\mathbb{Q}$. Furthermore, since we already proved WR-BR structure in discrete time, for all N and $s \in [t,T]$, the function $x \mapsto \Theta_s^{*N}(t,1,x,\kappa)$ is increasing. Due to the pathwise weak convergence as in (21), for all $s \in [t,T]$, the function $x \mapsto \Theta_s^*(t,1,x,\kappa)$ is increasing over rational x. In particular, $x \mapsto \Delta\Theta_t^*(t,1,x,\kappa) \equiv \Theta_{t+}^*(t,1,x,\kappa)$ is increasing over rational x. Since we only need to exclude the downward jump, it suffices to have this monotonicity over the rational numbers. □

5 Example models with WR-BR structure

By Theorem 4.2, any model satisfying Assumption 4.1 has WR-BR structure. In this section, we show that Assumption 4.1 is satisfied by several standard processes. We start with deterministic K.

Proposition 5.1. (Deterministic case).
Assume that $K\colon [0,T] \to (0,\infty)$ *is deterministic and two times continuously differentiable,* $\rho\colon [0,T] \to (0,\infty)$ *is continuously differentiable with* $K_t'+2\rho_t K_t > 0$ *for all* $t \in [0,T]$*. Then Assumption 4.1 holds, and the value function has WR-BR structure.*

Proof. Condition i) is equivalent to $K_t' + 2\rho_t K_t > 0$, and ii), iii) are clearly satisfied for deterministic continuous K. □

Let us now turn to a time-homogeneous geometric Brownian motion (GBM). Notice that, due to the homogeneity in time, it is enough to verify the conditions in Assumption 4.1 only under measures $\mathbb{P}_{0,\kappa}$.

Proposition 5.2. (GBM case).
Let K *be a geometric Brownian motion*

$$dK_t = \bar{\mu} K_t \, dt + \bar{\sigma} K_t \, dW_t^K, \quad K_0 = \kappa > 0.$$

Consider a constant resilience $\rho_t \equiv \bar{\rho} > 0$ *such that* $2\bar{\rho} + \bar{\mu} - \bar{\sigma}^2 > 0$*. Then Assumption 4.1 holds, and the value function has WR-BR structure.*

Proof. i) We have $\eta_t = \frac{1}{K_t}\left(2\bar{\rho} + \bar{\mu} - \bar{\sigma}^2\right) > 0$.

ii) Set $q_t := \frac{1}{K_t}$. Thanks to Hölder's inequality,

$$\mathbb{E}_{0,\kappa}\left[\frac{\left(\sup_{t\in[0,T]} K_t\right)^2}{\inf_{t\in[0,T]} K_t}\right] \le \mathbb{E}_{0,\kappa}\left[\sup_{t\in[0,T]} K_t^4\right]^{\frac{1}{2}} \mathbb{E}_{0,\kappa}\left[\sup_{t\in[0,T]} q_t^2\right]^{\frac{1}{2}}. \tag{22}$$

The explicit formula for GBM, $K_t = K_0 e^{\bar{\sigma} W_t^K + \left(\bar{\mu} - \frac{\bar{\sigma}^2}{2}\right)t}$, yields

$$\mathbb{E}_{0,\kappa}\left[\sup_{t\in[0,T]} K_t^4\right] \le \kappa^4 \max\left\{1, e^{4\left(\bar{\mu} - \frac{\bar{\sigma}^2}{2}\right)T}\right\} \mathbb{E}_{0,\kappa}\left[\exp\left(4\bar{\sigma} \sup_{t\in[0,T]} W_t^K\right)\right].$$

The latter expression is finite due to the fact that $(\sup_{t\in[0,T]} W_t^K)$ has the same distribution as $|W_T^K|$, which is a consequence of the reflection principle for a Brownian motion. The second expectation in (22) is finite, since $q_t = \frac{1}{K_t}$ is also a GBM (with drift $(\bar{\sigma}^2 - \bar{\mu})$ and volatility $\bar{\sigma}$).

iii) Due to the form of η_t, it is enough to consider

$$\mathbb{E}_{0,\kappa}\left[\int_0^T \left(\sup_{t\in[0,T]} K_t\right)^2 \frac{1}{K_t}\, dt\right] \le T\, \mathbb{E}_{0,\kappa}\left[\frac{\left(\sup_{t\in[0,T]} K_t\right)^2}{\inf_{t\in[0,T]} K_t}\right],$$

where the right-hand side is finite according to ii). □

See Fruth (2011) for alternative conditions ensuring WR-BR structure in the GBM case. We conclude this section with the Cox-Ingersoll-Ross (CIR) process. This process is particularly appealing from the economic point of view due to its mean reversion.[4]

[4] See Fruth (2011), Section 3.3, for numerical illustrations of WR-BR barriers, optimal trading strategies and cost distribution functions for K being a CIR process.

Proposition 5.3. (CIR case).
Let K be a Cox-Ingersoll-Ross process

$$dK_t = \bar{\mu}(\bar{K} - K_t)\,dt + \bar{\sigma}\sqrt{K_t}\,dW_t^K, \quad K_0 = \kappa > 0,$$

where $\bar{K}, \bar{\mu}, \bar{\sigma} > 0$. Consider a constant resilience $\rho_t \equiv \bar{\rho} > 0$ such that

$$2\bar{\rho} \geq \bar{\mu} > 2\bar{\sigma}^2/\bar{K}.$$

Then Assumption 4.1 holds, and the value function has WR-BR structure.

Proof. Such a CIR process stays a.s. strictly positive, as the Feller condition $\bar{\mu}\bar{K} \geq \bar{\sigma}^2/2$ is met. Moreover, it turns out that $\eta_t = \frac{1}{K_t}(2\bar{\rho}-\bar{\mu}) + \frac{1}{K_t^2}(\bar{\mu}\bar{K}-\bar{\sigma}^2) > 0$ due to our assumptions. Conditions ii) and iii) both hold by showing

$$\mathbb{E}_{0,\kappa}\left[\frac{\left(\sup_{t\in[0,T]} K_t\right)^2}{\left(\inf_{t\in[0,T]} K_t\right)^2}\right] < \infty.$$

Thanks to Hölder's inequality, with $q_t = \frac{1}{K_t}$, we have

$$\mathbb{E}_{0,\kappa}\left[\frac{\left(\sup_{t\in[0,T]} K_t\right)^2}{\left(\inf_{t\in[0,T]} K_t\right)^2}\right] \leq \mathbb{E}_{0,\kappa}\left[\sup_{t\in[0,T]} K_t^8\right]^{\frac{1}{4}} \mathbb{E}_{0,\kappa}\left[\sup_{t\in[0,T]} q_t^{\frac{8}{3}}\right]^{\frac{3}{4}}. \tag{23}$$

Since the drift of the CIR process is bounded above, we can isolate the local martingale part of K and use the Burkholder-Davis-Gundy inequalities.[5] With appropriate positive constants $\bar{c}_n$, we obtain

$$\begin{aligned}\mathbb{E}_{0,\kappa}\left[\sup_{t\in[0,T]} K_t^8\right] &\leq \bar{c}_1\left\{\kappa^8 + \left(\bar{\mu}\bar{K}T\right)^8 + \mathbb{E}_{0,\kappa}\left[\sup_{t\in[0,T]}\left|\int_0^t \bar{\sigma}\sqrt{K_s}\,dW_s^K\right|^8\right]\right\} \qquad (24)\\ &\leq \bar{c}_2\left\{\kappa^8 + \left(\bar{\mu}\bar{K}T\right)^8 + \mathbb{E}_{0,\kappa}\left[\left(\int_0^T \bar{\sigma}^2 K_s\,ds\right)^4\right]\right\}.\end{aligned}$$

The latter expectation is finite because all positive moments of the CIR process are finite (see, e.g., Filipovic and Mayerhofer (2009)).

It remains to show that the second term on the right-hand side of (23) is finite. By Itô's formula, the process $q_t = \frac{1}{K_t}$ has the dynamics

$$dq_t = \left(\bar{\mu}q_t - \left(\bar{\mu}\bar{K} - \bar{\sigma}^2\right)q_t^2\right)\,dt - \bar{\sigma}q_t^{\frac{3}{2}}\,dW_t^K.$$

With these preparations, we proceed similarly to (24):

$$\begin{aligned}\mathbb{E}_{0,\kappa}\left[\sup_{t\in[0,T]} q_t^{\frac{8}{3}}\right] &\leq \bar{c}_3\left\{\kappa^{-\frac{8}{3}} + \left(\frac{\bar{\mu}^2 T}{4\left(\bar{\mu}\bar{K} - \bar{\sigma}^2\right)}\right)^{\frac{8}{3}} + \mathbb{E}_{0,\kappa}\left[\sup_{t\in[0,T]}\left|\int_0^t \bar{\sigma}q_s^{\frac{3}{2}}\,dW_s^K\right|^{\frac{8}{3}}\right]\right\}\\ &\leq \bar{c}_4\left\{\kappa^{-\frac{8}{3}} + \left(\frac{\bar{\mu}^2 T}{4\left(\bar{\mu}\bar{K} - \bar{\sigma}^2\right)}\right)^{\frac{8}{3}} + \mathbb{E}_{0,\kappa}\left[\left(\int_0^T \bar{\sigma}^2 q_s^3\,ds\right)^{\frac{4}{3}}\right]\right\}.\end{aligned}$$

[5] For every $m > 0$, there exist universal positive constants k_m and K_m such that

$$k_m\mathbb{E}\left[\langle M\rangle_\tau^m\right] \leq \mathbb{E}\left[\left(\max_{t\leq\tau}|M_t|\right)^{2m}\right] \leq K_m\mathbb{E}\left[\langle M\rangle_\tau^m\right]$$

for every continuous local martingale M with $M_0 = 0$ and every stopping time τ. See, e.g., Karatzas and Shreve (2000), Chapter 3, Theorem 3.28.

We are done, since $\mathbb{E}_{0,\kappa}\left[\left(\int_0^T q_s^3\,ds\right)^{\frac{4}{3}}\right] \leq \bar{c}_5 \int_0^T \mathbb{E}_{0,\kappa}[q_s^4]\,ds$, and the fourth moment of the inverse CIR process is finite whenever $\bar{\mu}\bar{K} > 2\bar{\sigma}^2$ (see, e.g., Ahn and Gao (1999) for an explicit calculation of negative moments of the CIR process). $\square$

6 Example models without WR-BR structure

In this section, we provide examples that do not follow the WR-BR structure. In particular, we show that cases of WR-BR-WR structure can occur: when a large number of shares remains to be purchased, we may find that it is optimal to wait in spite of buying being optimal if a smaller number of shares is remaining. We first consider a tractable model with two scenarios and thereafter provide numerical results for a CIR model in discrete time. All of our examples are in discrete time with trading occurring at three points in time. The following proposition establishes that WR-BR structure always applies if trading occurs at only two points in time.

Proposition 6.1. (WR-BR structure for two trading instances).
Let $N = 1$, i.e. $0 = t_0 < t_1 = T$, and denote $a_0 := e^{-\int_{t_0}^{t_1} \rho_s ds}$. Then the value function has WR-BR structure with

$$\begin{aligned} V^{dis}(t_0, y, \kappa) &= \frac{1}{2}\mathbb{E}_{t_0,\kappa}[K_T]y^2 + a_0 y - \left\{ \begin{array}{ll} \frac{\left[\left(\mathbb{E}_{t_0,\kappa}[K_T]-\kappa a_0\right)y-(1-a_0)\right]^2}{2\kappa+2\mathbb{E}_{t_0,\kappa}[K_T]-4\kappa a_0} & \text{if } y > c(t_0,\kappa) \\ 0 & \text{otherwise} \end{array} \right\}, \\ c(t_0, \kappa) &= \left\{ \begin{array}{ll} \frac{1-a_0}{\mathbb{E}_{t_0,\kappa}[K_T]-\kappa a_0} & \text{if } \mathbb{E}_{t_0,\kappa}[K_T] > \kappa a_0 \\ \infty & \text{otherwise} \end{array} \right\}. \end{aligned}$$

Proof. We know that $U^{dis}(t_1, \delta, x, \kappa) = (\delta + \frac{\kappa}{2}x)x$. The assertion follows from

$$U^{dis}(t_0, \delta, x, \kappa) = \min_{\xi \in [0,x]} \left\{ \left(\delta + \frac{\kappa}{2}\xi\right)\xi + \mathbb{E}_{t_0,\kappa}\left[U^{dis}\left(t_1, (\delta + \kappa\xi)a_0, x - \xi, K_T\right)\right] \right\}.$$

$\square$

Note that we have not made any specific assumptions on the distribution of K_T in the proof of Proposition 6.1.

6.1 A model with two scenarios in discrete time

Let us assume that the process K is not driven by a diffusion, but instead is given by a finite number of scenarios. The case of a single scenario implies a deterministic evolution of K which always results in a WR-BR structure. We therefore focus on the second simplest case of two equally likely scenarios A and B, i.e. $\Omega = \{\omega_A, \omega_B\}$, and consider three trading instances $\{t_0, t_1, t_2\}$, i.e. $N = 2$. To fully specify this *two scenario model*, we need to choose seven constants

$$a_0 := e^{-\int_{t_0}^{t_1} \rho_s ds}, a_1 := e^{-\int_{t_1}^{t_2} \rho_s ds}, \kappa_0, \kappa_1^A := K_{t_1}(\omega_A), \kappa_2^A := K_{t_2}(\omega_A), \kappa_1^B := K_{t_1}(\omega_B), \kappa_2^B := K_{t_2}(\omega_B).$$

Proposition 6.2. *With the parameter values given in Figure 1, the optimal strategy is of WR-BR-WR structure, i.e., there are two threshold values $0 < c_u < c_l < \infty$ such that the buy region at time t_0 is given by $Br_{t_0} = (c_u, c_l]$.*

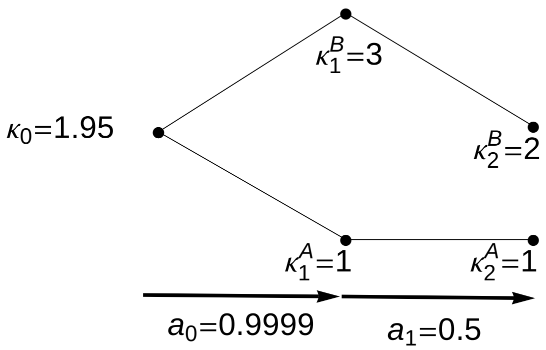

Figure 1: Seven constants that specify the two scenario model with three trading instances.

Proof. The optimal strategy is determined by ξ_0, ξ_1^A and ξ_1^B. Since $c(t_1,\kappa_1^A) = c(t_1,\kappa_1^B) = 1 =: c(t_1)$ by Proposition 6.1, we see that $\xi_1^A > 0$ if and only if $\xi_1^B > 0$.

Let us now consider a given trade ξ_0 at time t_0 and assume optimal trading thereafter. This results in a cost of

$$\tilde{U}^{dis}(t_0,\delta,x,\kappa_0;\xi_0) := \left(\delta + \frac{\kappa_0}{2}\xi_0\right)\xi_0 + \mathbb{E}\left[U^{dis}(t_1,(\delta+\kappa_0\xi_0)a_0, x-\xi_0,\kappa_1)\right].$$

It is easy to see that $\tilde{U}^{dis}$ is piecewise quadratic in ξ_0. For the section of ξ_0 where the optimal ξ_1 is positive ($\xi_1 > 0$), a straightforward calculation shows that the quadratic coefficient is negative. $\tilde{U}^{dis}$ therefore cannot attain its minimum in the interior of this section; the optimal strategy therefore satisfies $\xi_0 = 0$, $\xi_1 = 0$ or $\xi_0 = \xi_1 = 0$.

Using Proposition 6.1, we easily calculate that for trading only at times t_0 and t_2, we have

$$c_l := c^{0,2}(t_0) < 1/a_0 = c(t_1)/a_0.$$

Hence $(c_l, 1/a_0]$ must be a subset of the buy region Br_{t_0}. For $y > c(t_1)/a_0 = 1/a_0$, we need to compare the cost $U^{0,2}$ of optimally trading only at times t_0 and t_2 with the cost $U^{1,2}$ of optimally trading only at time t_1 and t_2. Using the parameter values given in Figure 1, we find that the quadratic coefficient of $U^{0,2}$ is larger than the quadratic coefficient of $U^{1,2}$; therefore there must be an intersection point $c_u > c_l$ where $U^{1,2} = U^{0,2}$. We then have for $y \le c_l$ that $U^{1,2} = U^{0,2}$ and the optimal strategy trades neither at t_0 nor t_1, for $c_l < y < c_u$ that $U^{0,2} < U^{1,2}$ and the unique optimal strategy trades at t_0 but not at t_1, for $y = c_u$ that $U^{0,2} = U^{1,2}$ and there are two optimal strategies (one trading at t_0 but not t_1, and one trading at t_1 but not t_0), and for $y > c_u$ that $U^{0,2} > U^{1,2}$ and the unique optimal strategy trades at t_1 but not at t_0. ☐

To illustrate the dynamics of the optimal strategy, we take different x and plot

$$\xi \mapsto \tilde{U}^{dis}\left(t_0, 1, x, 1.95; \xi\right)$$

in Figure 2. When the total order is as small as $x = 0.9$, it is optimal not to do an initial trade. The transition from wait to buy region is approximately at $x = 0.95$. For $x = 1$, we are in the buy region and one optimally trades about two percent of the total order at time t_0. But at $x = 5.75$, we switch from buy to wait region and stay in the wait region for all larger values of x. The graph for $x = 5.75$ illustrates the *non-uniqueness* of the optimal strategy at the transition from buy to wait region.

Intuition might suggest that the larger the remaining position x at time t_0, the larger the initial trade ξ_0. The downside of trading at time t_0 is that the full initial impact δ is influencing the cost functional (at later points in time this initial impact is partially decayed already). The upside is a more balanced distribution of new impact across an additional time point (any impact generated at time t_0 will already be partially decayed at time t_1). These two effects are the only drivers in the case of deterministic K, and the second effect grows faster in the remaining position x than the first effect. If K evolves stochastically, then a third effect comes into play: trading at times after t_0 can respond to new information gained about K (such as whether scenario A or B occurred). This effect can dominate the second effect for large remaining positions x.

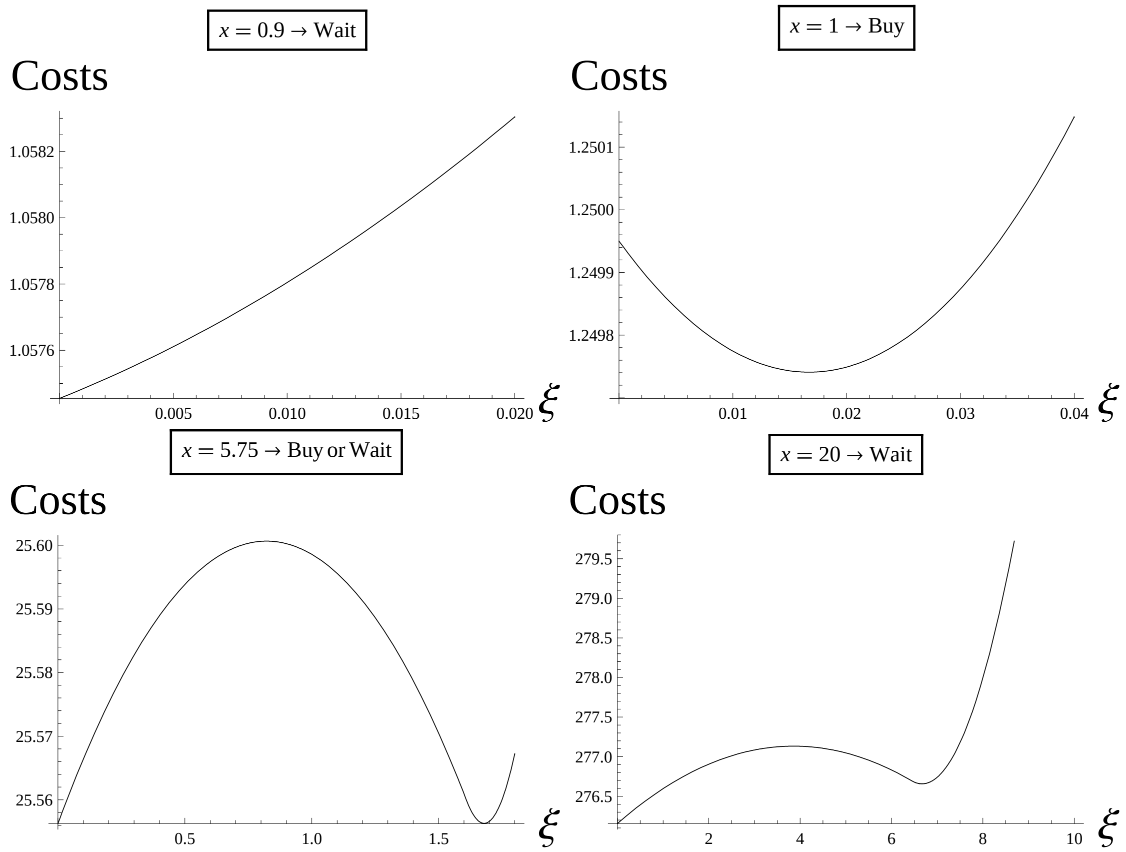

Figure 2: For the parameters from Figure 1 and total order size $x = 0.9,\ 1,\ 5.75,\ 20$, the graphs plot the dependence of the costs $\tilde{U}^{dis}\left(t_0, 1, x, 1.95; \xi\right)$ on the initial trade ξ.

Let us now analyze the situation for different values of κ_0 while keeping the other model parameters including κ_1^A, κ_1^B, κ_2^A and κ_2^B fixed. Figure 3 indicates for each point (κ_0, x) if it belongs to the buy or wait region. It is created by computing the optimal initial trade $\xi(\kappa_0, x)$ of $\tilde{U}^{dis}\left(t_0, 1, x, \kappa_0; \xi\right)$ analytically. WR-BR-WR structure occurs for $\kappa_0 \in (1.94, 2)$. The upper barrier from buy to wait region has an asymptote at $\kappa_0 = 1.94$. For the case $\kappa_0 = 1.95$ that we discussed in Figure 2, the small dots on the right-hand side of Figure 3 point out the transitions from wait to buy region and buy to wait region respectively. For expensive $\kappa_0 \geq 2$, we are not trading irrespectively of the size of the total order. For inexpensive $\kappa_0 \leq 1.94$, we have the usual WR-BR situation. On the interval in between, the large investor has an incentive not to trade for large positions x. The resilience between t_0 and t_1 is extremely low and waiting until t_1 has the advantage of gaining information whether scenario A or B has occurred. That is there is a tradeoff between gaining information by waiting until the next time instance and attracting resilience by trading right now.

The two scenario model in this section can be approximated by a diffusive model that has almost half of its scenarios arbitrarily close to scenario A and almost another half arbitrarily close to scenario B. For such a diffusive model it can be shown that it does not exhibit a WR-BR structure.

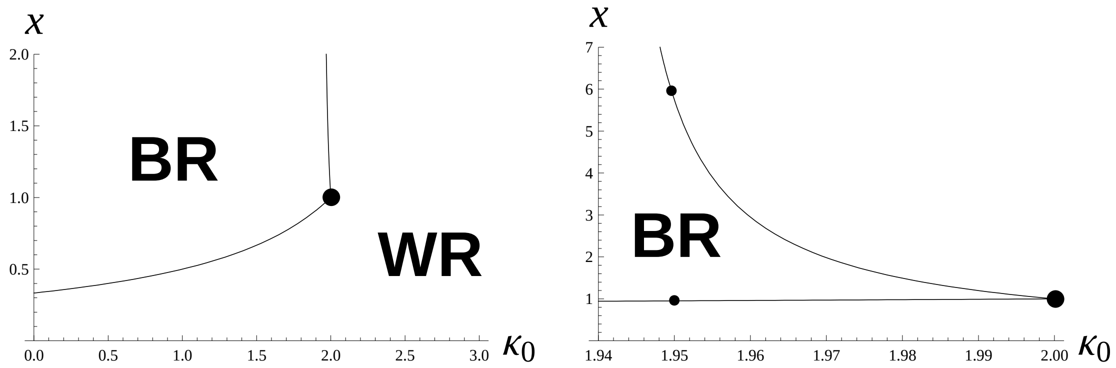

Figure 3: For the parameters from Figure 1, but different values of κ_0, we illustrate the wait and buy region. Looking more closely at the large dot $(\kappa_0, x) = (2, 1)$ yields the picture on the right-hand side. The buy region has the shape of a wedge.

6.2 Cox-Ingersoll-Ross process in discrete time

In Section 5, we have considered examples of diffusive models and shown that they have WR-BR structure if certain conditions are met. For the case of the CIR process[6]

$$dK_s = \bar{\mu}\left(\bar{K} - K_s\right) ds + \bar{\sigma}\sqrt{K_s} dW_s^K,$$

let us now consider three trading times $\{t_0, t_1, t_2\}$ with

$$\begin{aligned} &t_0 = 0,\ t_1 = 0.0072,\ t_2 = 1.0072,\ \rho \equiv 1.3863, \\ &\bar{\mu} = 0.6931,\ \bar{K} = 1,\ \bar{\sigma} = 5.2523. \end{aligned} \tag{25}$$

This example violates the conditions of Proposition 5.3. It is inspired by the two scenario model of the previous subsection. E.g., t_1 is close to t_0, and the high volatility makes illiquid scenarios with $K_t >> \bar{K}$ likely to occur.

Using Proposition 6.1 and the density function of the CIR process together with a numerical integration scheme, we can compute $\tilde{U}^{dis}(t_0, 1, x, \kappa_0; \xi)$ from dynamic programming. For each point (κ_0, x), we can calculate the costs for different trades ξ_0 from an equidistant grid $\{0, d\xi, ..., x\}$. We can then infer that the point (κ_0, x) belongs to the wait region if the costs for $\xi = 0$ are smaller than the costs on the remaining grid.

Executing this scheme for several points (κ_0, x) yields Figure 4. As for the two scenario model, there exist choices of κ_0 that lead to WR-BR-WR structure. But instead of a wedge-shaped buy region, we get a tongue-shaped upper wait region, which is located around the mean-reversion level $\bar{K} = 1$.

7 Profitable round trip strategies

So far we have only considered one side of the limit order book. In this section, we extend our model and include the other side of the limit order book. In such two-sided limit order books, round trip strategies are possible and we determine under which conditions they can be profitable.

Without loss of generality we now consider the starting time 0. We model strategies that both buy and sell the asset as a pair $(\Theta, \tilde{\Theta})$, where $\Theta \in \mathcal{A}_0$ and $\tilde{\Theta} \in \mathcal{A}_0$ describe the number of shares which

[6]See Fruth (2011) for a WR-BR-WR example for the time-inhomogeneous GBM and three trading instances.

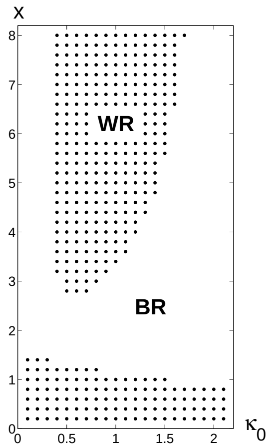

Figure 4: This figure shows a WR-BR-WR example for the CIR process with parameters (25) and three trading instances. Points $(\kappa_0, x) \in \{0.1, 0.2, ..., 2.1\} \times \{0.2, 0.4, ..., 8\}$ are considered. The wait region is shaded black.

the investor bought respectively sold starting from time 0. The position at time t is given by $\Theta_t - \tilde{\Theta}_t$, and a *round trip strategy* is characterised by $\Theta_{T+} = \tilde{\Theta}_{T+}$. Recall that, by definition of $\mathcal{A}_0$, Θ_{T+} and $\tilde{\Theta}_{T+}$ are bounded random variables. If A_t and B_t are the best ask and best bid prices respectively, then the total cost of a strategy $(\Theta, \tilde{\Theta})$ is given by

$$\mathcal{C}(\Theta, \tilde{\Theta}) := \int_{[0,T]} \left(A_t + \frac{K_t}{2} \Delta\Theta_t \right) d\Theta_t - \int_{[0,T]} \left(B_t - \frac{K_t}{2} \Delta\tilde{\Theta}_t \right) d\tilde{\Theta}_t. \tag{26}$$

We now present two different models for two-sided limit order books. The corresponding models for deterministic K are discussed in Fruth, Schöneborn, and Urusov (2014). First, we consider a two-sided limit order book with bid-ask spread that depends on trading activity.

Model 7.1. (Dynamic spread model).
The best ask and best bid price processes A and B in (26) are modelled as $A_t := A^u_t + D_t$ and $B_t := B^u_t - E_t$, where the *unaffected* best ask and best bid price processes A^u and B^u are càdlàg $\mathcal{H}^1$-martingales with $B^u_t \le A^u_t$ for all $t \in [0,T]$, and

$$D_t := D_0 e^{-\int_0^t \rho_s ds} + \int_{[0,t)} K_s e^{-\int_s^t \rho_u du} d\Theta_s, \quad t \in [0, T+], \tag{27}$$

$$E_t := E_0 e^{-\int_0^t \rho_s ds} + \int_{[0,t)} K_s e^{-\int_s^t \rho_u du} d\tilde{\Theta}_s, \quad t \in [0, T+], \tag{28}$$

with some given non-negative initial price impacts $D_0 \ge 0$ and $E_0 \ge 0$.

Proposition 7.2. (Profitable round trips in the dynamic spread model).
In the dynamic spread model round trip trading strategies cannot be profitable. That is, for all $\kappa > 0$, $D_0 \ge 0$ and $E_0 \ge 0$, for all admissible $(\Theta, \tilde{\Theta})$ with $\Theta_{T+} = \tilde{\Theta}_{T+}$, we have

$$\mathbb{E}_{0,\kappa}[\mathcal{C}(\Theta, \tilde{\Theta})] \ge 0.$$

Furthermore, the expected execution costs of a buy (or sell) program that builds up a deterministic position of say $x \in \mathbb{R}$ shares cannot be decreased by intermediate sell (resp. buy) trades. That is, for all $\kappa > 0$, $D_0 \ge 0$ and $E_0 \ge 0$, for any admissible $(\Theta, \tilde{\Theta})$ with $\Theta_{T+} - \tilde{\Theta}_{T+} = x > 0$, there is an admissible $\hat{\Theta}$ with $\hat{\Theta}_{T+} = x$ such that $\mathbb{E}_{0,\kappa}[\mathcal{C}(\Theta, \tilde{\Theta})] \ge \mathbb{E}_{0,\kappa}[\mathcal{C}(\hat{\Theta}, 0)]$; also the symmetric statement with $x < 0$ holds true.

We omit the proof since it is a direct extension of the corresponding Proposition 3.4 in Fruth, Schöneborn, and Urusov (2014). Let us now consider an alternative model for a two-sided limit order book in which the spread is constantly zero.

Model 7.3. (Zero spread model).
The best ask and best bid price processes in (26) are modelled as $A_t^\updownarrow := B_t^\updownarrow := S_t^u + D_t^\updownarrow$, where the *unaffected* price S^u is a càdlàg $\mathcal{H}^1$-martingale, and

$$D_t^\updownarrow := D_0^\updownarrow e^{-\int_0^t \rho_s\,ds} + \int_{[0,t)} K_s e^{-\int_s^t \rho_u\,du}(d\Theta_s - d\tilde{\Theta}_s), \quad t \in [0, T+], \tag{29}$$

with some given initial price impact $D_0^\updownarrow \in \mathbb{R}$.

There is a subtle difference in understanding price manipulation between the dynamic and zero spread models. In the discussion of profitable round trip strategies in the dynamic spread model (see Proposition 7.2) we considered arbitrary initial values $D_0 \geq 0$ and $E_0 \geq 0$ in (27) and (28). In contrast to this, in the discussion of profitable round trip strategies in the zero spread model (Theorem 7.4, which follows) we will consider $D_0^\updownarrow = 0$ in (29). Whenever $D_0^\updownarrow \neq 0$ we usually have profitable round trip strategies in the zero spread model, and this is due not to properties of the model, but rather to the fact that both buy and sell orders are executed at the same price, such that profitable round trips will make use of the initial deviation $D_0^\updownarrow$ from the unaffected price S^u and of the fact that, due to the resilience, the absolute value of this deviation decreases to zero in the absence of trading.[7]

In order to study profitable round trip strategies in the zero spread model let us introduce the notations

$$\Theta^\updownarrow := \Theta - \tilde{\Theta}$$

for the composite strategy, which includes both buy and sell orders, and, by analogy with (5),

$$J^\updownarrow(\Theta^\updownarrow) := J^\updownarrow(t, \delta, \Theta^\updownarrow, \kappa) := J_T^\updownarrow(t, \delta, \Theta^\updownarrow, \kappa) := \mathbb{E}_{t,\delta,\kappa}\left[\int_{[t,T]} \left(D_s^\updownarrow + \frac{K_s}{2}\Delta\Theta_s^\updownarrow\right) d\Theta_s^\updownarrow\right]$$

for the cost function.[8] We will sometimes write $J_T^\updownarrow$ with the subscript T to emphasize the time horizon explicitly. As in (5), the subscript in $\mathbb{E}_{t,\delta,\kappa}$ means that we start at time t with $D_t^\updownarrow = \delta$ and $K_t = \kappa$. Let us consider diffusion setting (14) for all finite time horizons $T < \infty$ and introduce the function $\eta\colon \mathbb{R}_+ \times (0, \infty) \to \mathbb{R}$ by the formula

$$\eta(s, \kappa) := \frac{2\rho_s}{\kappa} + \frac{\mu(s, \kappa)}{\kappa^2} - \frac{\sigma^2(s, \kappa)}{\kappa^3},$$

that is, we have $\eta_s = \eta(s, K_s)$ for η_s as in Assumption 4.1 i).

[7] See Remark 8.2 in Fruth, Schöneborn, and Urusov (2014) for more detail on this point.

[8] The precise explanation of how this formula comes into play is similar to the explanation in Footnote 1 on page 4. Namely, consider a strategy $(\Theta, \tilde{\Theta}) \in \mathcal{A}_t \times \mathcal{A}_t$ that acquires $\Theta_{T+}^\updownarrow = x$ shares on the time interval $[t, T]$ ($x \in \mathbb{R}$ is deterministic). The total cost of this strategy is (cf. (26))

$$\int_{[t,T]} \left(S_s^u + D_s^\updownarrow + \frac{K_s}{2}\Delta\Theta_s\right) d\Theta_s - \int_{[t,T]} \left(S_s^u + D_s^\updownarrow - \frac{K_s}{2}\Delta\tilde{\Theta}_s\right) d\tilde{\Theta}_s.$$

A calculation involving integration by parts and using that S^u is an $\mathcal{H}^1$-martingale as well as that Θ and $\tilde{\Theta}$ are bounded reveals that the expected total cost equals

$$S_t^u x + \mathbb{E}_{t,\delta,\kappa}\left[\int_{[t,T]} \left(D_s^\updownarrow + \frac{K_s}{2}\Delta\Theta_s\right) d\Theta_s - \int_{[t,T]} \left(D_s^\updownarrow - \frac{K_s}{2}\Delta\tilde{\Theta}_s\right) d\tilde{\Theta}_s\right].$$

Again, the first summand, which is trivial and moreover vanishes for round trip strategies, describes the expected cost that occurs due to trading in the unaffected price. The second summand in the latter formula, which describes the expected liquidity cost, is in general larger than $J^\updownarrow(t, \delta, \Theta^\updownarrow, \kappa)$, but it is equal to $J^\updownarrow(t, \delta, \Theta^\updownarrow, \kappa)$ whenever $\Theta_{T+} + \tilde{\Theta}_{T+}$ equals the variation of $\Theta^\updownarrow$ over $[t, T]$. It remains to notice that the latter can always be assumed without loss of generality (and, moreover, it does not make sense economically to consider strategies $(\Theta, \tilde{\Theta})$ with $\Theta_{T+} + \tilde{\Theta}_{T+}$ being strictly greater than the variation of $\Theta^\updownarrow$ over $[t, T]$ because this means that buying and selling happen simultaneously).

Theorem 7.4. (Profitable round trips in the zero spread model).
In the zero spread model suppose that Assumption 4.1 ii) holds for all finite $T < \infty$ and

(A) the resilience is bounded away from zero ($\rho_s \geq \bar{\rho} > 0$) as well as, for all $t \geq 0$ and $\kappa > 0$, the function $s \mapsto \mathbb{E}_{t,\kappa}[K_s]$, $s \in [t, \infty)$, is bounded.

We then have the following classification:

1. *If $\eta \geq 0$ everywhere, then all round trip strategies starting at any time $t \geq 0$ with $D_t^{\updownarrow} = 0$ have nonnegative costs.*

2. *Under Assumption 4.1 iii), if $\eta < 0$ in some $[t, t + \Delta t] \times [\kappa - \epsilon, \kappa + \epsilon]$, then there are profitable round trip strategies starting at t with $D_t^{\updownarrow} = 0$.*[9]

Assumption (A) is satisfied for a wide range of processes K including stationary processes such as the CIR process as well as the GBM process with non-positive drift ($\bar{\mu} \leq 0$) whenever the resilience is bounded away from zero.

We will see in the proof that the role of Assumption (A) is to ensure that liquidation of a random but bounded position that the investor has at some time $t+\Delta t$ can be achieved for arbitrarily small cost if $D^{\updownarrow}_{t+\Delta t+} = 0$ and the time horizon T is large. The latter property seems natural to expect in reasonable models when the resilience is bounded away from zero. In fact, one might replace Assumption (A) with any other assumption that ensures the property stated above.

Parts ii) and iii) of Assumption 4.1 are required for some technical aspects of our proof. That is, the main message of Theorem 7.4 can be somewhat loosely described as follows: if $\eta \geq 0$ everywhere, then round trip strategies cannot be profitable; if $\eta < 0$ somewhere,[10] then profitable round trip strategies exist.

Proof. We can extend the proof of (15) to the zero spread model and find that the cost function $J^{\updownarrow}$ satisfies

$$J^{\updownarrow}(t, \delta, \Theta^{\updownarrow}, \kappa) = \frac{1}{2}\mathbb{E}_{t,\delta,\kappa}\left[\frac{(D^{\updownarrow}_{T+})^2}{K_T} - \frac{\delta^2}{\kappa} + \int_{[t,T]} \eta_s (D^{\updownarrow}_s)^2\, ds\right] \tag{30}$$

with $\eta_s \equiv \eta(s, K_s)$ as in Assumption 4.1 i). More precisely, instead of monotone convergence in (16) we need to use dominated convergence, which applies because Θ and $\tilde{\Theta}$ are bounded and $\mathbb{E}_{t,\kappa}[\sup_{s\in[t,T]} K_s] < \infty$ (the latter follows from Assumption 4.1 ii)), and again dominated convergence works in (17) (based on Assumption 4.1 ii)). As for (18), we use monotone convergence in the first case ($\eta \geq 0$), while dominated convergence applies in the second case (due to Assumption 4.1 iii)). In particular, when we start at time t with $D_t^{\updownarrow} = 0$, we have

$$J^{\updownarrow}(t, 0, \Theta^{\updownarrow}, \kappa) = \frac{1}{2}\mathbb{E}_{t,0,\kappa}\left[\frac{(D^{\updownarrow}_{T+})^2}{K_T} + \int_{[t,T]} \eta_s (D^{\updownarrow}_s)^2\, ds\right],$$

which establishes the statement in the first case ($\eta \geq 0$ everywhere).

Similarly to (30) we establish that, for any stopping time τ with $t \leq \tau \leq T$, it holds

$$J^{\updownarrow}_T(t, \delta, \Theta^{\updownarrow}, \kappa) = \frac{1}{2}\mathbb{E}_{t,\delta,\kappa}\left[\frac{(D^{\updownarrow}_{\tau+})^2}{K_\tau} - \frac{\delta^2}{\kappa} + \int_{[t,\tau]} \eta_s (D^{\updownarrow}_s)^2\, ds\right] + \mathbb{E}_{t,\delta,\kappa}\left[\int_{(\tau,T]} \left(D^{\updownarrow}_s + \frac{K_s}{2}\Delta\Theta^{\updownarrow}_s\right) d\Theta^{\updownarrow}_s\right]. \tag{31}$$

[9] As pointed out above profitable round trip strategies exist also for $D_t^{\updownarrow}$ being different from zero, but the relevant question in the zero spread model is the one for $D_t^{\updownarrow} = 0$.

[10] Let us also notice that, if $\eta(t, \kappa) < 0$ at some point (t, κ) and the functions ρ, μ and σ are continuous, then $\eta < 0$ in some $[t, t + \Delta t] \times [\kappa - \epsilon, \kappa + \epsilon]$.

We now make use of (31) to construct a profitable round trip strategy in the second case. Starting at (t,κ) with $D^{\updownarrow}_t = 0$, let us define the stopping time

$$\tau := (t+\Delta t) \wedge \inf\{s \geq t \mid K_s \notin (\kappa-\epsilon, \kappa+\epsilon)\}$$

and consider the following trading strategy. First, buy $x > 0$ units of the asset at time t. This results in $D^{\updownarrow}_{t+} = \kappa x$. At time τ, we have $D^{\updownarrow}_\tau = \kappa x e^{-\int_t^\tau \rho_s\,ds}$ and sell $y = \frac{D^{\updownarrow}_\tau}{K_\tau}$ units of the asset, resulting in $D^{\updownarrow}_{\tau+} = 0$. We do nothing in $(\tau, t+\Delta t)$ and then liquidate the position $x-y$ with a uniform speed between $t+\Delta t$ and T. Notice that the position $x-y$ is random (it depends on K_τ), but bounded (due to the construction of τ). Summarizing, we consider the following round trip strategy: $\Theta^{\updownarrow}_t = 0$, $\Theta^{\updownarrow}_s = x$ for $s \in (t,\tau]$, $\Theta^{\updownarrow}_s = x-y$ for $s \in (\tau, t+\Delta t]$,

$$\Theta^{\updownarrow}_s = x - y + \frac{s-t-\Delta t}{T-t-\Delta t}\,(y-x), \quad s \in (t+\Delta t, T],$$

and $\Theta^{\updownarrow}_{T+} = \Theta^{\updownarrow}_T = 0$. An application of (31) in this case yields

$$J^{\updownarrow}_T(t,0,\Theta^{\updownarrow},\kappa) = \frac{1}{2}\mathbb{E}_{t,0,\kappa}\left[\int_{[t,\tau]} \eta_s (D^{\updownarrow}_s)^2\,ds\right] + \mathbb{E}_{t,0,\kappa}\left[\int_{(t+\Delta t,T]} D^{\updownarrow}_s\,d\Theta^{\updownarrow}_s\right].$$

The first term on the right-hand side is strictly negative (we are considering the second case) and does not depend on T. Below we present a calculation showing that the second term goes to zero as T goes to infinity, which means that, for a sufficiently large T, we constructed a round trip strategy with strictly negative cost, i.e. with strictly positive profit.

Relying on Assumption (A) we finally show that

$$\mathbb{E}_{t,0,\kappa}\left[\int_{(t+\Delta t,T]} D^{\updownarrow}_s\,d\Theta^{\updownarrow}_s\right] \xrightarrow[T\to\infty]{} 0 \tag{32}$$

for the strategy described above. Recall that $D^{\updownarrow}_{\tau+} = 0$, hence $D^{\updownarrow}_{t+\Delta t+} = 0$. That is, for $s \in (t+\Delta t, T]$, we have

$$D^{\updownarrow}_s = \frac{y-x}{T-t-\Delta t}\int_{t+\Delta t}^s K_u e^{-\int_u^s \rho_r\,dr}\,du,$$

therefore,

$$\begin{aligned}
\mathbb{E}_{t,0,\kappa}\left[\int_{(t+\Delta t,T]} D^{\updownarrow}_s\,d\Theta^{\updownarrow}_s\right] &= \mathbb{E}_{t,\kappa}\left[\frac{(y-x)^2}{(T-t-\Delta t)^2}\int_{t+\Delta t}^T\int_{t+\Delta t}^s K_u e^{-\int_u^s \rho_r\,dr}\,du\,ds\right] \\
&\leq \frac{\text{const}}{(T-t-\Delta t)^2}\,\mathbb{E}_{t,\kappa}\left[\int_{t+\Delta t}^T\int_{t+\Delta t}^s K_u e^{-\int_u^s \rho_r\,dr}\,du\,ds\right],
\end{aligned} \tag{33}$$

where we used that the random variable $(y-x)^2$ is bounded. Further,

$$\begin{aligned}
\mathbb{E}_{t,\kappa}\left[\int_{t+\Delta t}^T\int_{t+\Delta t}^s K_u e^{-\int_u^s \rho_r\,dr}\,du\,ds\right] &= \int_{t+\Delta t}^T \left(\int_u^T e^{-\int_u^s \rho_r\,dr}\,ds\right)\mathbb{E}_{t,\kappa}[K_u]\,du \\
&\leq \frac{1}{\rho}\left[\int_{t+\Delta t}^T \mathbb{E}_{t,\kappa}[K_u]\,du\right] \leq \text{const}\,(T-t-\Delta t).
\end{aligned}$$

Together with (33), we obtain (32). This completes the proof. □

The results of this section reveal a link between the models for two-sided limit order books: If Assumption 4.1 i) holds, then optimal strategies in the dynamic spread model are of WR-BR structure and profitable round trip strategies do not exist in the zero spread model. If Assumption 4.1 i) is violated, then optimal strategies in the dynamic spread model do not need to be of WR-BR structure, and round trip strategies in the zero spread model do not need to result in costs.

If only deterministic trading strategies are considered, then only the expected evolution of K matters and, in the case $\mu(s,\kappa)$ is affine in κ, $\tilde{\eta}_s := \frac{2\rho_s}{K_s} + \frac{\mu(s,K_s)}{K_s^2} \geq 0$ is sufficient to prevent free (or even profitable) deterministic round trip strategies. Since $\eta_s = \tilde{\eta}_s - \frac{\sigma^2(s,K_s)}{K_s^3} < \tilde{\eta}_s$, we can have $\tilde{\eta}_s \geq 0$ while $\eta_s < 0$. For some stochastic models for K, we therefore have only stochastic profitable round trip strategies but no deterministic profitable round trip strategies.

8 Conclusion

We propose a limit order book model with stochastic liquidity that captures random fluctuations of the limit order book depth. If the stochastic liquidity in this model follows a diffusion process meeting certain conditions, then optimal trade execution follows the classical wait region / buy region structure often observed in limit order book models with static or deterministically time dependent liquidity. For other stochastic liquidity processes, the optimal trade execution strategy can take more general forms; for example, multiple wait regions can occur, and optimal trade sizes do not need to depend monotonically on the size of the position that remains to be liquidated. The conditions for the wait region / buy region structure also result in all round trip strategies generating positive costs even if the zero spread model is assumed.

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